Soluble two-species diffusion-limited models in arbitrary dimensions

M. Mobilia and P.-A. Bares
Institute of Theoretical Physics, Swiss Federal Institute of Technology of Lausanne, CH-1015 Lausanne EPFL, Switzerland
(Received 17 July 2000; published 27 February 2001)

A class of two-species (*three-states*) bimolecular diffusion-limited models of classical particles with hard-core reacting and diffusing in a hypercubic lattice of arbitrary dimension is investigated. The manifolds on which the equations of motion of the correlation functions close, are determined explicitly. This property allows to solve for the density and the two-point (two-time) correlation functions in arbitrary dimension for both, a translation invariant class and another one where translation invariance is broken. Systems with correlated as well as uncorrelated, yet random initial states can also be treated exactly by this approach. We discuss the asymptotic behavior of density and correlation functions in the various cases. The dynamics studied is very rich.

DOI: 10.1103/PhysRevE.63.036121 PACS number(s): 02.50.Ey, 05.50.+q, 82.20.Db, 82.40.Bj

I. INTRODUCTION

Nonequilibrium statistical mechanics has witnessed recently a resurgence of interest. Though over 50 years old, the field is still in its infancy. Powerful concepts and tools are being developed, and yet much progress remains to be done. The understanding of classical stochastic many-body systems is of relevance to a wide class of phenomena in physics and beyond. In this context, a class of models describing diffusion-limited reactions plays an important role [1]. The natural language to describe the stochastic dynamics of N classical bodies is that of the master equation. Formally, the dynamics can be coded in an imaginary time Schrödinger equation, where the Markov generator plays the role of the Hamiltonian (see, e.g., Refs. [1,2] and references therein). In the past, various representations in terms of spins, fermions or bosons have been used depending on the physics being emphasized. A powerful method that relies on bosonic field theory and the renormalization group has been applied by Cardy and collaborators to deal with low density systems in arbitrary dimensions (see, e.g., Ref. [3]). In one spatial dimension alternative approaches have been proposed which take into account the hard core of the classical particles (see, e.g., Refs. [1,2] and references therein). Since the pioneering work of Glauber on the stochastic Ising model [4], various generalizations and extensions have appeared (e.g., Refs. [5,6]) . A general approach has been proposed by Schütz [7]. The latter investigates the most general class of single-species models of bimolecular diffusion-limited reactions that can be solved exactly. Upon imposing constraints on the available manifold, the equations of motion of all correlation functions close and, in that sense, the dynamics is completely soluble. Via a duality transformation, Schütz further shows that on the 10-parametric soluble manifold, the spectrum of the stochastic Hamiltonian coincides with that of the XXZ-Heisenberg model (see also Ref. [8]). In Ref. [9], Fujii and Wadati extend Schütz's ideas to the s-species models of bimolecular diffusion-limited reaction processes. These authors derive the general constraints that allow for the equations of motion of correlation functions to close and, similarly to the single species case, introduce a dual Hamiltonian with identical spectrum. They further note that in the general multispecies case, the constraints of solubility (in the sense given above) do not seem to imply a simple relationship to integrable quantum Hamiltonians.

The general s-species bimolecular reaction–diffusion processes are characterized by $(s+1)^4-(s+1)^2$ independent parameters (reaction-rates) and we have to impose $2s^3$ constraints to close the hierarchy of the equations of motion of correlation functions. As few exact and explicit results for the *dynamics* of multispecies processes are available (in particular in dimensions $d>1$, see the discussion at the end of Sec. III), we decided to investigate in some details and generality the soluble two-species bimolecular diffusion-limited reaction systems. In this paper, we focus on the two-species problem $(s=2)$ and obtain, in arbitrary dimension, exact results. A particular physical application of this work to a *three-states* growth model will be presented elsewhere [10].

The paper is organized as follows: the remainder of this section will be devoted to definitions and notations. In Sec. II, the equations of motion of the density and two-point correlation functions are derived. The constraints that ensure the solubility of the problem are explicitly identified. We classify the soluble manifolds which will be investigated in the sequel. In the first part of Sec. III, we study the Fourier–Laplace transform of the density in the soluble case (on a 56-parameters manifold). In the second part, we compute on two manifolds the exact expression of the density in arbitrary dimensions. We provide the asymptotic behavior of the latter for three different initial conditions. At the end of Sec. III we discuss the relationship between our results and the solution of some models solved exactly in dimension $d\geqslant 1$ [11–16]. In the first part of Sec. IV, we give the exact dynamic form factor for an homogeneous and uncorrelated (yet random) initial state. In the second part, we compute, in arbitrary dimension, the exact two-time two-point correlation functions for random (homogeneous) uncorrelated as well as correlated initial states (we discuss the sensitiveness of the system to the presence of initial correlations). Section V is devoted to the study of the instantaneous two-points correlation functions on a manifold of translationally invariant models. We first deal with the one-dimensional case, which is investigated for random uncorrelated as well as correlated initial states. Further, we consider the higher dimensional case with random (yet homogeneous) and uncorrelated initial states. The last section, is devoted to the conclusion.

For clarity and brevity's sake, some definitions, as well as some technical details, are given in the appendixes.

Consider a hypercubic lattice of dimension d with periodic boundary conditions and N sites ($N=L^d$), where L represents the linear dimension of the hypercube. Further assume that local bimolecular reactions between species A and B take place. Each site is either empty (denoted by the symbol 0) or occupied at most by one particle of type A (respectively, B) denoted in the following by the index 1 (respectively, 2). The dynamics is parametrized by the transition rates $\Gamma_{\beta_1\beta_2}^{\beta_3\beta_4}$, where $\beta_1,\beta_2,\beta_3,\beta_4=0,1,2$:

$$\forall(\beta_1,\beta_2)\neq(\beta_3,\beta_4), \quad \Gamma_{\beta_1\beta_2}^{\beta_3\beta_4}: \quad \beta_1+\beta_2\rightarrow\beta_3+\beta_4. \tag{1}$$

Probability conservation implies

$$\Gamma_{\beta_1\beta_2}^{\alpha_1\beta_2}=-\sum_{(\beta_1,\beta_2)\neq(\beta_1',\beta_2')}\Gamma_{\beta_1\beta_2}^{\beta_1'\beta_2'} \tag{2}$$

with

$$\Gamma_{\beta_1\beta_2}^{\beta_3\beta_4}\geqslant 0, \quad \forall(\beta_1,\beta_2)\neq(\beta_3,\beta_4). \tag{3}$$

For example the rate Γ_{11}^{12} corresponds to the process $A+A\rightarrow A+B$, while conservation of probability leads to $\Gamma_{11}^{11}=-(\Gamma_{11}^{10}+\Gamma_{11}^{01}+\Gamma_{11}^{00}+\Gamma_{11}^{02}+\Gamma_{11}^{20}+\Gamma_{11}^{21}+\Gamma_{11}^{12}+\Gamma_{11}^{22})$.

The state of the system is determined by specifying the probability for the occurrence of configuration $\{n\}$ at time t. It is represented by the ket

$$|P(t)\rangle=\sum_{\{n\}}P(\{n\},t)|\{n\}\rangle, \tag{4}$$

where the sum runs over the 3^N configurations ($N=L^d$). At site i the local state is denoted by the ket $|n_i\rangle=(100)^T$ if the site i is empty, $|n_i\rangle=(010)^T$ if the site i is occupied by a particle of type A (1) and $|n_i\rangle=(001)^T$ otherwise. It is by now well known that the master equation governing the dynamics of the systems can be rewritten as an imaginary-time Schrödinger equation:

$$\frac{\partial}{\partial t}|P(t)\rangle=-H|P(t)\rangle, \tag{5}$$

where H is the Markov generator, also called *stochastic Hamiltonian*, which in general is neither Hermitian nor normal. The construction of the *stochastic Hamiltonian H* from the master equation follows a known procedure (see, e.g., Refs. [1,2,7,9]). We define [1,2,7,9] the *left vacuum* $\langle\tilde{\chi}|$:

$$\langle\tilde{\chi}|\equiv\sum_{\{n\}}\langle\{n\}|. \tag{6}$$

Probability conservation yields the local equation (stochasticity of H)

$$\langle\tilde{\chi}|H=\sum_{\alpha=1,\dots,d}\sum_m\langle\tilde{\chi}|H_{m,m+e^\alpha}=0\Rightarrow\langle\tilde{\chi}|H_{m,m+e^\alpha}=0, \tag{7}$$

where $e^\alpha, 1\leqslant\alpha\leqslant d$, designates, in Cartesian coordinates, the unit vector along the α direction.

The two-species local Markov generator acts on two adjacent sites, i.e.,

$$-H_{m,m+e^\alpha}=\begin{pmatrix}
\Gamma_{00}^{00} & \Gamma_{01}^{00} & \Gamma_{02}^{00} & \Gamma_{10}^{00} & \Gamma_{11}^{00} & \Gamma_{12}^{00} & \Gamma_{20}^{00} & \Gamma_{21}^{00} & \Gamma_{22}^{00}\\
\Gamma_{00}^{01} & \Gamma_{01}^{01} & \Gamma_{02}^{01} & \Gamma_{10}^{01} & \Gamma_{11}^{01} & \Gamma_{12}^{01} & \Gamma_{20}^{01} & \Gamma_{21}^{01} & \Gamma_{22}^{01}\\
\Gamma_{00}^{02} & \Gamma_{01}^{02} & \Gamma_{02}^{02} & \Gamma_{10}^{02} & \Gamma_{11}^{02} & \Gamma_{12}^{02} & \Gamma_{20}^{02} & \Gamma_{21}^{02} & \Gamma_{22}^{02}\\
\Gamma_{00}^{10} & \Gamma_{01}^{10} & \Gamma_{02}^{10} & \Gamma_{10}^{10} & \Gamma_{11}^{10} & \Gamma_{12}^{10} & \Gamma_{20}^{10} & \Gamma_{21}^{10} & \Gamma_{22}^{10}\\
\Gamma_{00}^{11} & \Gamma_{01}^{11} & \Gamma_{02}^{11} & \Gamma_{10}^{11} & \Gamma_{11}^{11} & \Gamma_{12}^{11} & \Gamma_{20}^{11} & \Gamma_{21}^{11} & \Gamma_{22}^{11}\\
\Gamma_{00}^{12} & \Gamma_{01}^{12} & \Gamma_{02}^{12} & \Gamma_{10}^{12} & \Gamma_{11}^{12} & \Gamma_{12}^{12} & \Gamma_{20}^{12} & \Gamma_{21}^{12} & \Gamma_{22}^{12}\\
\Gamma_{00}^{20} & \Gamma_{01}^{20} & \Gamma_{02}^{20} & \Gamma_{10}^{20} & \Gamma_{11}^{20} & \Gamma_{12}^{20} & \Gamma_{20}^{20} & \Gamma_{21}^{20} & \Gamma_{22}^{20}\\
\Gamma_{00}^{21} & \Gamma_{01}^{21} & \Gamma_{02}^{21} & \Gamma_{10}^{21} & \Gamma_{11}^{21} & \Gamma_{12}^{21} & \Gamma_{20}^{21} & \Gamma_{21}^{21} & \Gamma_{22}^{21}\\
\Gamma_{00}^{22} & \Gamma_{01}^{22} & \Gamma_{02}^{22} & \Gamma_{10}^{22} & \Gamma_{11}^{22} & \Gamma_{12}^{22} & \Gamma_{20}^{22} & \Gamma_{21}^{22} & \Gamma_{22}^{22}
\end{pmatrix}, \tag{8}$$

where the same notations as in Ref. [9] have been used. Probability conservation implies that each column in the-above representation sums up to zero. Locally, the left vacuum $\langle\tilde{\chi}|$ has the representation

$$\langle\tilde{\chi}|=(111)\otimes(111)=(111111111). \tag{9}$$

The action of any operator on the left vacuum has then a simple summation interpretation. This observation will be crucial in the following computations. Below we shall assume an initial state $|P(0)\rangle$ and investigate the expectation value of an operator O (observables such as density, etc.),

$$\langle O\rangle(t)\equiv\langle\tilde{\chi}|Oe^{-Ht}|P(0)\rangle. \tag{10}$$

II. DYNAMICAL EQUATIONS OF MOTION

Exploiting the properties of the left vacuum $\langle\tilde{\chi}|$ and denoting by $n_i^{\beta},\beta\in\{0,1,2\}$ the occupation of site i by a particle of type β, we derive below the equations of motion of the density and two-point correlation function. For $\beta=0$, n_i^0 denotes the empty state at site i, i.e.,

$$n_i^0=\mathbb{1}-n_i^A-n_i^B. \tag{11}$$

We evaluate the action of H on the operators n_i^A, n_i^B, taking into account the local nature of the Markov generator:

$$-\langle\tilde{\chi}|n_m^A H_{m,m\pm e^{\alpha}},\quad -\langle\tilde{\chi}|n_m^B H_{m,m\pm e^{\alpha}}. \tag{12}$$

As an example, the first term in the above yields

$$\begin{aligned}
-\langle\tilde{\chi}|n_m^A H_{m,m+e^{\alpha}} &= \sum_{\gamma,\delta=0,1,2}\{(\Gamma^{10}_{\gamma\delta}+\Gamma^{11}_{\gamma\delta}+\Gamma^{12}_{\gamma\delta})\langle\tilde{\chi}|n_m^{\gamma}n_{m+e^{\alpha}}^{\delta}\}\\
&=\langle\tilde{\chi}|(\Gamma^{10}_{00}+\Gamma^{11}_{00}+\Gamma^{12}_{00})+[(\Gamma^{10}_{10}+\Gamma^{11}_{10}+\Gamma^{12}_{10})-(\Gamma^{10}_{00}+\Gamma^{11}_{00}+\Gamma^{12}_{00})]\langle\tilde{\chi}|n_m^A\\
&\quad+[(\Gamma^{10}_{01}+\Gamma^{11}_{01}+\Gamma^{12}_{01})-(\Gamma^{10}_{00}+\Gamma^{11}_{00}+\Gamma^{12}_{00})]\langle\tilde{\chi}|n_{m+e^{\alpha}}^A+[(\Gamma^{10}_{20}+\Gamma^{11}_{20}+\Gamma^{12}_{20})\\
&\quad-(\Gamma^{10}_{00}+\Gamma^{11}_{00}+\Gamma^{12}_{00})]\langle\tilde{\chi}|n_m^B+[(\Gamma^{10}_{02}+\Gamma^{11}_{02}+\Gamma^{12}_{02})-(\Gamma^{10}_{00}+\Gamma^{11}_{00}+\Gamma^{12}_{00})]\langle\tilde{\chi}|n_{m+e^{\alpha}}^B\\
&\quad+[(\Gamma^{10}_{00}+\Gamma^{11}_{00}+\Gamma^{12}_{00})-(\Gamma^{10}_{01}+\Gamma^{11}_{01}+\Gamma^{12}_{01})+(\Gamma^{10}_{11}+\Gamma^{11}_{11}+\Gamma^{12}_{11})-(\Gamma^{10}_{10}+\Gamma^{11}_{10}+\Gamma^{12}_{10})]\langle\tilde{\chi}|n_m^A n_{m+e^{\alpha}}^A\\
&\quad+[(\Gamma^{10}_{00}+\Gamma^{11}_{00}+\Gamma^{12}_{00})-(\Gamma^{10}_{02}+\Gamma^{11}_{02}+\Gamma^{12}_{02})+(\Gamma^{10}_{22}+\Gamma^{11}_{22}+\Gamma^{12}_{22})-(\Gamma^{10}_{20}+\Gamma^{11}_{20}+\Gamma^{12}_{20})]\langle\tilde{\chi}|n_m^B n_{m+e^{\alpha}}^B\\
&\quad+[(\Gamma^{10}_{00}+\Gamma^{11}_{00}+\Gamma^{12}_{00})-(\Gamma^{10}_{02}+\Gamma^{11}_{02}+\Gamma^{12}_{02})+(\Gamma^{10}_{12}+\Gamma^{11}_{12}+\Gamma^{12}_{12})-(\Gamma^{10}_{10}+\Gamma^{11}_{10}+\Gamma^{12}_{10})]\langle\tilde{\chi}|n_m^A n_{m+e^{\alpha}}^B\\
&\quad+[(\Gamma^{10}_{00}+\Gamma^{11}_{00}+\Gamma^{12}_{00})-(\Gamma^{10}_{01}+\Gamma^{11}_{01}+\Gamma^{12}_{01})+(\Gamma^{10}_{21}+\Gamma^{11}_{21}+\Gamma^{12}_{21})-(\Gamma^{10}_{20}+\Gamma^{11}_{20}+\Gamma^{12}_{20})]\langle\tilde{\chi}|n_m^B n_{m+e^{\alpha}}^A,
\end{aligned} \tag{13}$$

where the use of Eq. (2) is required to substitute for $\Gamma^{11}_{11},\Gamma^{10}_{10},\Gamma^{12}_{12}$. As expected, the stochastic Hamiltonian connects the one-body initial operator to a two-body expression.

The equation of motion for the density becomes (at site m)

$$\begin{aligned}
\frac{d}{dt}\langle n_m^{A,B}\rangle(t)&\equiv\frac{d}{dt}\langle\tilde{\chi}|n_m^{A,B}e^{-Ht}|P(0)\rangle\\
&=-\sum_{e^{\alpha}}\langle n_m^{A,B}(H_{m,m+e^{\alpha}}+H_{m-e^{\alpha},m})\rangle(t).
\end{aligned} \tag{14}$$

In order to determine the second moments, we also need to evaluate the following terms:

$$\begin{aligned}
&-\langle\tilde{\chi}|n_m^A n_{m+e^{\alpha}}^A H_{m,m+e^{\alpha}},\quad -\langle\tilde{\chi}|n_m^A n_{m+e^{\alpha}}^B H_{m,m+e^{\alpha}},\\
&-\langle\tilde{\chi}|n_m^B n_{m+e^{\alpha}}^B H_{m,m+e^{\alpha}};\quad -\langle\tilde{\chi}|n_m^A n_{m+e^{\alpha}}^B H_{m,m+e^{\alpha}}.
\end{aligned} \tag{15}$$

For the sake of illustration, the first term above yields

$$\begin{aligned}
&-\langle\tilde{\chi}|n_m^A n_{m+e^{\alpha}}^A H_{m,m+e^{\alpha}}\\
&\quad=\Gamma^{11}_{00}\langle\tilde{\chi}|+(\Gamma^{11}_{10}-\Gamma^{11}_{00})\langle\tilde{\chi}|n_m^A+(\Gamma^{11}_{01}-\Gamma^{11}_{00})\langle\tilde{\chi}|n_{m+e^{\alpha}}^A\\
&\qquad+(\Gamma^{11}_{20}-\Gamma^{11}_{00})\langle\tilde{\chi}|n_m^B+(\Gamma^{11}_{02}-\Gamma^{11}_{00})\langle\tilde{\chi}|n_{m+e^{\alpha}}^B\\
&\qquad+(\Gamma^{11}_{00}+\Gamma^{11}_{11}-\Gamma^{01}_{11}-\Gamma^{10}_{11})\langle\tilde{\chi}|n_m^A n_{m+e^{\alpha}}^A\\
&\qquad+(\Gamma^{11}_{00}+\Gamma^{11}_{22}-\Gamma^{11}_{02}-\Gamma^{11}_{20})\langle\tilde{\chi}|n_m^B n_{m+e^{\alpha}}^B\\
&\qquad+(\Gamma^{11}_{00}+\Gamma^{11}_{12}-\Gamma^{11}_{02}-\Gamma^{11}_{10})\langle\tilde{\chi}|n_m^A n_{m+e^{\alpha}}^B\\
&\qquad+(\Gamma^{11}_{00}+\Gamma^{11}_{21}-\Gamma^{11}_{20}-\Gamma^{11}_{01})\langle\tilde{\chi}|n_m^B n_{m+e^{\alpha}}^A.
\end{aligned} \tag{16}$$

Notice that the evolution operator connects a two-body operator to a two-body expression.

To compute the two-point correlation functions, we have to distinguish the sites that are nearest neighbors from those that are not. If the sites m and n are not nearest neighbors $[\mathrm{dist}(m,n)>1]$, the equation of motion reads

$$
\begin{aligned}
-\frac{d}{dt}\langle n_m^{A,B} n_n^{A,B}\rangle(t) = \sum_{\alpha} & [\langle (n_m^{A,B} H_{m-e^{\alpha},m}) n_n^{A,B}\rangle(t) \\
& + \langle n_m^{A,B}(n_n^{A,B} H_{n-e^{\alpha},n})\rangle(t)] \\
& + \sum_{\alpha} [\langle (n_m^{A,B} H_{m,m+e^{\alpha}}) n_n^{A,B}\rangle(t) \\
& + \langle n_m^{A,B}(n_n^{A,B} H_{n,n+e^{\alpha}})\rangle(t)]. \qquad (17)
\end{aligned}
$$

while if the sites are nearest neighbors, we have

$$
\begin{aligned}
& -\frac{d}{dt}\langle n_m^{A,B} n_{m+e^{\alpha}}^{A,B}\rangle(t) \\
& = \langle n_m^{A,B} n_{m+e^{\alpha}}^{A,B} H_{m,m+e^{\alpha}}\rangle(t) \\
& \quad + \sum_{\alpha'} [\langle (n_m^{A,B} H_{m-e^{\alpha'},m}) n_{m+e^{\alpha}}^{A,B}\rangle(t) \\
& \quad + \langle n_m^{A,B}(n_{m+e^{\alpha}}^{A,B} H_{m+e^{\alpha},m+e^{\alpha}+e^{\alpha'}})\rangle(t)] \\
& \quad + \sum_{\alpha'\neq\alpha} [\langle (n_m^{A,B} H_{m,m+e^{\alpha'}}) n_{m+e^{\alpha}}^{A,B}\rangle(t) \\
& \quad + \langle n_m^{A,B}(n_{m+e^{\alpha}}^{A,B} H_{m+e^{\alpha}-e^{\alpha'},m+e^{\alpha}})\rangle(t)]. \qquad (18)
\end{aligned}
$$

The equations of motion of n-points correlation functions are obtained in a similar way, with help of (12) and (13) and (15) and (16). As is well known, the equations of motion of classical (or quantum) correlation functions constitute an open hierarchy which is not soluble in general. However, if we impose on the $3^4-3^2=72$ bimolecular transition rates involving two adjacent sites, the following 16 constraints [see Appendix A for the definitions (A1) and (A2)]:

$$
\begin{aligned}
&(1)\ A_2^a+A_1^a+A_0^a=\Gamma_{11}^{10}+\Gamma_{11}^{11}+\Gamma_{11}^{12},\\
&(2)\ B_2^a+B_1^a+A_0^a=\Gamma_{22}^{10}+\Gamma_{22}^{11}+\Gamma_{22}^{12},\\
&(3)\ B_2^a+A_1^a+A_0^a=\Gamma_{12}^{10}+\Gamma_{12}^{11}+\Gamma_{12}^{12},\\
&(4)\ A_2^a+B_1^a+A_0^a=\Gamma_{21}^{10}+\Gamma_{21}^{11}+\Gamma_{21}^{12},\\
&(5)\ C_2^a+C_1^a+C_0^a=\Gamma_{11}^{01}+\Gamma_{11}^{11}+\Gamma_{11}^{21},\\
&(6)\ D_2^a+D_1^a+C_0^a=\Gamma_{22}^{01}+\Gamma_{22}^{11}+\Gamma_{22}^{21},\\
&(7)\ D_2^a+C_1^a+C_0^a=\Gamma_{12}^{01}+\Gamma_{12}^{11}+\Gamma_{12}^{21},\\
&(8)\ C_2^a+D_1^a+C_0^a=\Gamma_{21}^{01}+\Gamma_{21}^{11}+\Gamma_{21}^{21},\\
&(9)\ A_2^b+A_1^b+A_0^b=\Gamma_{11}^{20}+\Gamma_{11}^{21}+\Gamma_{11}^{22},\\
&(10)\ B_2^b+B_1^b+A_0^b=\Gamma_{22}^{20}+\Gamma_{22}^{21}+\Gamma_{22}^{22},\\
&(11)\ B_2^b+A_1^b+A_0^b=\Gamma_{12}^{20}+\Gamma_{12}^{21}+\Gamma_{12}^{22},\\
&(12)\ A_2^b+A_0^b+B_1^b=\Gamma_{21}^{20}+\Gamma_{21}^{21}+\Gamma_{21}^{22},\\
&(13)\ C_0^b+C_1^b+C_2^b=\Gamma_{11}^{02}+\Gamma_{11}^{12}+\Gamma_{11}^{22},\\
&(14)\ C_0^b+D_1^b+D_2^b=\Gamma_{22}^{02}+\Gamma_{22}^{12}+\Gamma_{22}^{22},\\
&(15)\ C_0^b+C_1^b+D_2^b=\Gamma_{12}^{02}+\Gamma_{12}^{12}+\Gamma_{12}^{22},\\
&(16)\ C_0^b+C_2^b+D_1^b=\Gamma_{21}^{02}+\Gamma_{21}^{12}+\Gamma_{21}^{22},
\end{aligned}
\qquad (19)
$$

the equations of motion of the density and two-point correlation functions (and all multipoints correlation functions) become closed. It is worth emphasizing that when the hierarchy closes at the lowest level, i.e., at the level of the density, the equations of motion of *all* higher correlation functions also close. This is a remarkable property. In the expressions (18) and (19), the rates $\Gamma_{\alpha\beta}^{\alpha\beta}$ have not been made explicit for brevity.

A general diffusion-limited two-species reaction model is defined on the manifold, $V_{\mathrm{par}}=\{\Gamma_{\alpha\beta}^{\gamma\delta}-\{\Gamma_{\alpha\beta}^{\alpha\beta}\}|\alpha,\beta\in(0,1,2)\}$, which has here $3^4-9=72$ independent parameters. Let us denote by V_{sol} the restriction of V_{par} on the $(72-16=56$ parameters) manifold defined by the additional constraints (19): $V_{\mathrm{sol}}\equiv V_{\mathrm{par}}\cap(19)$. The latter represents the manifold on which the equations of motion of the correlation functions are closed, i.e., the soluble manifold. We can further require translation invariance, i.e., $\langle n_m^i n_{m+|r|}^j\rangle(t)=\langle n_0^i n_{|r|}^j\rangle(t)\equiv\mathcal{G}_{|r|}^{ij},\forall r,t\ \ [i,j\in(A,B)]$ and in particular $\langle n_m^A n_n^B\rangle(t)=\langle n_m^B n_n^A\rangle(t)\equiv\mathcal{G}_{|n-m|}^{AB}$. Imposing the above conditions in equations (17) and (18) and taking into account the conditions of solubility (19), we arrive at the manifold $V_{\mathrm{transl-invar}}$, the restriction of V_{sol} on the translation invariant soluble dynamics. Notice that $V_{\mathrm{transl-invar}}(d)=V_{\mathrm{sol}}\cap V'(d)$, where $V'(d)=\{E_0^{ab}=E_0^{ba};\quad F_1^{ab}+F_2^{ab}+A_0^b d+C_0^b(d-1)=F_1^{ba}+F_2^{ba}+C_0^b d+A_0^b(d-1);\ F_3^{ab}+F_4^{ab}+A_0^a(d-1)+C_0^a d=F_3^{ba}+F_4^{ba}+C_0^a(d-1)+A_0^a d;\quad H_1^{ab}+H_2^{ab}+(C_2^a+B_1^b)d+(A_1^a+D_2^b)(d-1)=H_1^{ba}+H_2^{ba}+(A_1^a+D_2^b)d+(B_1^b+C_2^a)(d-1);\ A_2^a+D_1^b=B_2^b+C_1^a;\ G_1^{ba}+C_2^b d+A_1^d(d-1)=G_1^{ab}+C_2^b(d-1)+A_1^b d;\ \ G_2^{ab}+B_a^a(d-1)+D_2^a d=G_2^{ba}+B_1^a d+D_2^a(d-1);\ B_2^a=D_1^a;\ \ A_2^b=C_1^b.\}$ Therefore this manifold has $72-16-9=47$ independent parameters. In practice, however further constraints may be required for the computations to be accessible. With this remark in mind, we define the manifolds $V''=\{A_1^b+C_2^b=A_2^b=C_1^b=B_1^a+D_2^a=B_2^a=D_1^a=0\}$ and $V'''=\{A_n^b=B_n^a=C_n^b=D_n^a=G_2^a=G_1^b=G_1^{ab}=G_2^{ab}=H_n^{a,b}=0\quad|n=1,2\}$.

Summarizing the cases that we will discuss in this paper:

(i) For the case where translation invariance is broken, we shall compute the exact density on the manifold V_1, which has $72-16-6=50$ independent parameters,

$$V_1\equiv\cap V_{\mathrm{sol}}\cap V''. \qquad (20)$$

(ii) For the translation invariant case, we shall evaluate both, the density and two-point correlation function exactly on the manifold $V_2(d)$,

$$V_2(d)\equiv V_{\mathrm{sol}}\cap V'(d)\cap V'''\equiv V_{\mathrm{transl-invar}}(d)\cap V''' \qquad (21)$$

which has $47-16=31$ independent parameters.

To conclude this section, it is worth noting that there are few cases in which the open hierarchy of equations of motion can be solved analytically. The class of single-species one-dimensional models for which the evolution operator can be cast into a *free fermionic* form is an important example. However, the procedure of free ''fermionization'' cannot be applied to higher dimensions and/or multispecies problems, contrary to the method followed here.

III. THE DENSITY: GENERAL DISCUSSION

In the first part of this section we compute exactly the Fourier–Laplace transform of the density on the 56-parameters manifold $V_{\rm sol}$, which is, as are correlation functions, directly related to light scattering measurements in *real* reaction–diffusion systems [17–19]. The computation on the most general soluble manifold is here manageable because the linear differential difference equations governing the dynamics give rise to a 2×2 matrix, the properties of which can be studied analytically. For higher order correlation functions and/or for the s-species case, with $s>2$, the problem is however technically much harder (we shall come back to the general case in a future work). In the second part of this section we provide the density of species A and B in space and time, both in the translation invariant case and in a situation where translation invariance is broken. On the manifold $(V_{\rm par}\cap V_{\rm sol})\supset(V_{\rm transl-invar}\cap V_{\rm sol})$, we have

$$\frac{d}{dt}\langle n_m^A\rangle=(A_0^a+C_0^a)d+\langle n_m^A\rangle(t)(A_1^a+C_2^a)d+\sum_\alpha\,[A_2^a\langle n^A_{m+e^\alpha}\rangle(t)+C_1^a\langle n^A_{m-e^\alpha}\rangle(t)]+\langle n_m^B\rangle(t)(B_1^a+D_2^a)d+\sum_\alpha\,[B_2^a\langle n^B_{m+e^\alpha}\rangle(t)+D_1^a\langle n^B_{m-e^\alpha}\rangle(t)], \tag{22}$$

$$\frac{d}{dt}\langle n_m^B\rangle=(A_0^b+C_0^b)d+\langle n_m^B\rangle(t)(B_1^b+D_2^b)d+\sum_\alpha\,[B_2^b\langle n^B_{m+e^\alpha}\rangle(t)+D_1^b\langle n^B_{m-e^\alpha}\rangle(t)]+\langle n_m^A\rangle(t)(A_1^b+C_2^b)d+\sum_\alpha\,[A_2^b\langle n^A_{m+e^\alpha}\rangle(t)+C_1^b\langle n^A_{m-e^\alpha}\rangle(t)]. \tag{23}$$

Let us first consider the most general soluble case which is characterized by the set of equations (22) and (23). The solution of (22) and (23) is split into the solution of the homogeneous system $\langle n_m^A\rangle_h(t)$ $(\langle n_m^B\rangle_h(t))$ and a function $f_A(t)$ $(f_B(t))$ that takes into account the inhomogeneity, i.e.,

$$\frac{d}{dt}f_A(t)=(A_0^a+C_0^a)d+f_A(t)(A_1^a+A_2^a+C_1^a+C_2^a)d+f_B(t)(B_1^a+B_2^a+D_1^a+D_2^a)d, \tag{24}$$

$$\frac{d}{dt}f_B(t)=(A_0^b+C_0^b)d+f_B(t)(B_1^b+B_2^b+D_1^b+D_2^b)d+f_A(t)(A_1^b+A_2^b+C_1^b+C_2^b)d. \tag{25}$$

We introduce the Fourier transforms of $\langle n_m^{A,B}\rangle_h(t)$, i.e.,

$$\langle n_m^{A,B}\rangle_h(t)=\sum_{\vec p\in 1BZ}\langle\hat n_{\vec p}^{A,B}\rangle(t)e^{i\vec p.m}\Leftrightarrow\langle\hat n_{\vec p}^{A,B}\rangle(t)=\frac{1}{L^d}\sum_m\,\langle n_m^{A,B}\rangle_h(t)e^{-i\vec p.m}, \tag{26}$$

where the sum on $\vec p$ runs over the first Brillouin zone (1.BZ). The solution of the homogeneous problem in Fourier space reads

$$\begin{pmatrix}\langle\hat n_{\vec p}^A\rangle(t)\\ \langle\hat n_{\vec p}^B\rangle(t)\end{pmatrix}=e^{\mathcal{M}(p)t}\begin{pmatrix}\langle\hat n_{\vec p}^A\rangle(t=0)\\ \langle\hat n_{\vec p}^B\rangle(t=0)\end{pmatrix}, \tag{27}$$

where $\mathcal{M}_{i,j}(p),(i,j)\in(1,2)$ is a $s\times s=2\times 2$ matrix with the entries

$$\begin{aligned}\mathcal{M}_{1,1}(p)&=(A_1^a+C_2^a)d+\sum_\alpha\,(A_2^ae^{i\vec p.e_\alpha}+C_1^ae^{-i\vec p.e_\alpha}),\\ \mathcal{M}_{1,2}(p)&=(B_1^a+D_2^a)d+\sum_\alpha\,(B_2^ae^{i\vec p.e_\alpha}+D_1^ae^{-i\vec p.e_\alpha}),\\ \mathcal{M}_{2,1}(p)&=(A_1^b+C_2^b)d+\sum_\alpha\,(A_2^be^{i\vec p.e_\alpha}+C_1^be^{-i\vec p.e_\alpha}),\\ \mathcal{M}_{2,2}(p)&=(B_1^b+D_2^b)d+\sum_\alpha\,(B_2^be^{i\vec p.e_\alpha}+D_1^be^{-i\vec p.e_\alpha}).\end{aligned} \tag{28}$$

The eigenvalues of the matrix $\mathcal{M}$, which represent the inverse relaxation times of the system, control the asymptotic behavior of the density,

$$\lambda_\pm(p)=\frac{\mathcal{M}_{1,1}(p)+\mathcal{M}_{2,2}(p)}{2}\pm\frac{\sqrt{[\mathcal{M}_{1,1}(p)-\mathcal{M}_{2,2}(p)]^2+4\mathcal{M}_{1,2}(p)\mathcal{M}_{2,1}(p)}}{2}. \tag{29}$$

It has been shown in considering the one-dimensional alternating-bonds Ising model obeying Glauber's dynamics [20] that the relaxational eigenvalues of the analog of the matrix $\mathcal{M}$ allow to identify the critical (but nonuniversal) behavior: it is determined by the long wavelength p modes of the analog of the acoustic λ_- branch. One-dimensional alternating-bonds ($J_1 > J_2 > 0$) Ising model, with Glauber's dynamics, exhibits a nonuniversal critical dynamical exponent $z = 1 + (J_1/J_2)$ [20].

In the sequel we shall need the zero-momentum 2×2 matrix $\mathcal{M}(p=0) \equiv \mathcal{M}(0)$,

$$\mathcal{M}_{1,1}(0) = (A_1^a + A_2^a + C_1^a + C_2^a)d,$$

$$\mathcal{M}_{1,2}(0) = (B_1^a + B_2^a + D_1^a + D_2^a)d,$$

$$\mathcal{M}_{2,1}(0) = (A_1^b + A_2^b + C_1^b + C_2^b)d,$$

$$\mathcal{M}_{2,2}(0) = (B_1^b + B_2^b + D_1^b + D_2^b)d,$$

whose eigenvalues we shall denote, for short,

$$\gamma_\pm = \lambda_\pm(p=0).$$

Notice that at $p=0$, $\mathrm{Tr}\,\mathcal{M}(0) < 0$ and $\det \mathcal{M}(0) \geqslant 0$.

We are now in a position to compute the Fourier–Laplace transform $\mathcal{S}_0^{A,B}(\vec{p},\omega)$ of the density: $\mathcal{S}_0^{A,B}(\vec{p},\omega) \equiv (1/L^d)\Sigma_m \int_0^\infty dt\, e^{-i\vec{p}.m - \omega t}\langle n_m^{A,B}\rangle(t)$.

We consider initial states $\langle \hat{n}_p^{A,B}\rangle(0) = (1/L^d)\Sigma_m \langle n_m^{A,B}\rangle(0)\, e^{-i\vec{p}.m}[\leftrightarrow \langle n_m^{A,B}\rangle(0) = \Sigma_{\vec{p}\in 1BZ}\langle \hat{n}_p^{A,B}\rangle(0)\, e^{i\vec{p}.m}]$ and assume that the matrix $\mathcal{M}(p)$ is regular [i.e., $\det \mathcal{M}(p) \neq 0$]. We shall distinguish four cases.

(1) We assume that the matrix $\mathcal{M}(p)$ is diagonalizable, i.e., $\lambda_+(p) \neq \lambda_-(p)$, but not triangular, $\mathcal{M}_{1,2}(p) \neq 0$, and obtain

$$\begin{aligned}\mathcal{S}_0^A(\vec{p},\omega) = {} & \frac{1}{\lambda_-(p)-\lambda_+(p)}\left[\frac{\lambda_-(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_+(p)} - \frac{\lambda_+(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_-(p)}\right]\langle \hat{n}_p^A(0)\rangle \\ & + \frac{\mathcal{M}_{1,2}(p)\langle \hat{n}_p^B(0)\rangle}{[\omega-\lambda_+(p)][\omega-\lambda_-(p)]} \\ & + \frac{\delta_{p,0}}{\gamma_- - \gamma_+}\{[\gamma_- - \mathcal{M}_{1,1}(0)](A_0^a + C_0^a)d \\ & - \mathcal{M}_{1,2}(0)(A_0^b + C_0^b)d\}\frac{1}{\omega(\omega-\gamma_+)} \\ & - \frac{\delta_{p,0}}{\gamma_- - \gamma_+}\{[\gamma_+ - \mathcal{M}_{1,1}(0)](A_0^a + C_0^a)d \\ & - \mathcal{M}_{1,2}(0)(A_0^b + C_0^b)d\}\frac{1}{\omega(\omega-\gamma_-)} \end{aligned} \tag{30}$$

and

$$\begin{aligned}\mathcal{S}_0^B(\vec{p},\omega) = {} & \langle \hat{n}_p^B(0)\rangle\left[\frac{\lambda_-(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_-(p)} - \frac{\lambda_+(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_+(p)}\right] \\ & - \left[\frac{[\lambda_+(p)-\mathcal{M}_{1,1}(p)][\lambda_-(p)-\mathcal{M}_{1,1}(p)]}{\mathcal{M}_{1,2}(p)(\omega-\lambda_+(p))(\omega-\lambda_-(p))}\right] \\ & \times\langle \hat{n}_p^A(0)\rangle + \frac{\delta_{p,0}}{\gamma_- - \gamma_+}\left[\frac{[\gamma_- - \mathcal{M}_{1,1}(0)]}{\mathcal{M}_{1,2}(0)}(A_0^a \right. \\ & \left. + C_0^a)d - (A_0^b + C_0^b)d\right]\frac{\gamma_+ - \mathcal{M}_{1,1}(0)}{\omega(\omega-\gamma_+)} \\ & - \frac{\delta_{p,0}}{\gamma_- - \gamma_+}\left[\frac{[\gamma_+ - \mathcal{M}_{1,1}(0)]}{\mathcal{M}_{1,2}(0)}(A_0^a + C_0^a)d \right. \\ & \left. - (A_0^b + C_0^b)d\right]\frac{\gamma_- - \mathcal{M}_{1,1}(0)}{\omega(\omega-\gamma_-)}. \end{aligned} \tag{31}$$

Notice that the inhomogeneous part of the equations of motion give rise to a zero-momentum contribution which we shall omit hereafter. As expected the poles in the ω plane occur at the relaxational eigenvalues.

(2) Next consider the case where the matrix $\mathcal{M}(p)$ is nondiagonalizable and nontriangular: $\lambda(p) = \lambda_+(p) = \lambda_-(p) = [\mathcal{M}_{1,1}(p) + \mathcal{M}_{2,2}(p)]/2$ and $\mathcal{M}_{1,1}(p) \neq \mathcal{M}_{2,2}(p)$. We can compute $e^{\mathcal{M}(p)t}$ with the help of a Jordan decomposition of the matrix $\mathcal{M}(p)$, namely,

$$e^{\mathcal{M}(p)t} = P(p)e^{\mathcal{M}'(p)t}P^{-1}(p), \tag{32}$$

where P is a regular 2×2 matrix and $\mathcal{M}'(p) = \mathcal{M}_1(p) + \mathcal{M}_2(p)$ is the sum of a diagonal matrix $\mathcal{M}_1$ and a Jordan-block matrix. $\mathcal{M}_2(p)$ is chosen such that $[\mathcal{M}_1(p), \mathcal{M}_2(p)] = 0$. $\mathcal{M}_2(p)$ is nilpotent $[(\mathcal{M}_2(p))^2 = 0]$. Thus,

$$\begin{aligned} e^{\mathcal{M}(p)t} &= P(p)e^{\mathcal{M}'(p)t}P(p)^{-1} \\ &= \begin{pmatrix}\alpha(p) & \epsilon(p)\\ \beta(p) & \delta(p)\end{pmatrix}\begin{pmatrix} e^{\lambda(p)t} & \lambda(p)te^{\lambda(p)t}\\ 0 & e^{\lambda(p)t}\end{pmatrix} \\ &\quad\times\begin{pmatrix}\delta(p) & -\epsilon(p)\\ -\beta(p) & \alpha(p)\end{pmatrix}\frac{1}{[\alpha(p)\delta(p)-\beta(p)\epsilon(p)]}, \end{aligned} \tag{33}$$

with

$$P(p) = \begin{pmatrix}\alpha(p) & \epsilon(p)\\ \beta(p) & \delta(p)\end{pmatrix} \tag{34}$$

which entries are

$$\alpha(p) \equiv \frac{\lambda(p)}{\mathcal{M}_{1,1}(p)}\left(1 + \frac{\mathcal{M}_{1,2}(p)\mathcal{M}_{2,1}(p)}{\det \mathcal{M}(p)}\right),$$

$$\beta(p) \equiv -\frac{\lambda(p)\mathcal{M}_{2,1}(p)}{\det \mathcal{M}(p)},$$

$$\epsilon(p) \equiv \frac{\lambda(p)[\mathcal{M}_{2,2}(p)-\mathcal{M}_{1,2}(p)]}{\det \mathcal{M}(p)},$$

$$\delta(p)\equiv\frac{\lambda(p)}{\mathcal{M}_{1,2}(p)}\left(1-\frac{\mathcal{M}_{1,1}(p)[\mathcal{M}_{2,2}(p)-\mathcal{M}_{1,2}(p)]}{\det\mathcal{M}(p)}\right). \tag{35}$$

The matrix $P(p)$ is regular $[\det P(p)\neq 0]$ if $\mathcal{M}(p)$ is regular. This decomposition leads to the form factors $(\vec{p}\neq 0)$.

$$\mathcal{S}_0^A(\vec{p},\omega)=\frac{1}{\omega-\lambda(p)}\bigg[\langle\hat{n}_p^A\rangle(0)-\frac{\lambda(p)}{[\omega-\lambda(p)]\det P(p)}\times[\alpha^2(p)\langle\hat{n}_p^B\rangle(0)-\alpha(p)\beta(p)\langle\hat{n}_p^A\rangle(0)]\bigg]. \tag{36}$$

$$\mathcal{S}_0^B(\vec{p},\omega)=\frac{1}{\omega-\lambda(p)}\bigg[\langle\hat{n}_p^B\rangle(0)-\frac{\lambda(p)}{[\omega-\lambda(p)]\det P(p)}\times[\alpha(p)\beta(p)\langle\hat{n}_p^B\rangle(0)-\beta^2(p)\langle\hat{n}_p^A\rangle(0)]\bigg]. \tag{37}$$

Notice that $\lambda_+(p)=\lambda_-(p)=\lambda(p)=[\mathcal{M}_{1,1}(p)+\mathcal{M}_{2,2}(p)]/2$, imply the following relations on the reaction rates:

$$\sum_{\beta,\beta'=0,1,2}(\Gamma_{10}^{\beta2}\Gamma_{20}^{\beta'1}+\Gamma_{00}^{\beta1}\Gamma_{00}^{\beta'2})\leqslant\sum_{\beta,\beta'=0,1,2}(\Gamma_{10}^{\beta2}\Gamma_{00}^{\beta'1}+\Gamma_{00}^{\beta2}\Gamma_{20}^{\beta'1}),$$

$$\sum_{\beta,\beta'=0,1,2}(\Gamma_{01}^{2\beta}\Gamma_{02}^{1\beta'}+\Gamma_{00}^{1\beta}\Gamma_{00}^{2\beta'})\leqslant\sum_{\beta,\beta'=0,1,2}(\Gamma_{00}^{2\beta}\Gamma_{02}^{1\beta'}+\Gamma_{00}^{1\beta}\Gamma_{01}^{2\beta'}),$$

$$\sum_{\beta,\beta'=0,1,2}[\Gamma_{20}^{1\beta}\Gamma_{10}^{2\beta'}+\Gamma_{20}^{1\beta}\Gamma_{01}^{\beta'2}+\Gamma_{02}^{\beta1}\Gamma_{01}^{\beta'2}+\Gamma_{02}^{\beta1}\Gamma_{10}^{2\beta'}+(\Gamma_{00}^{1\beta}+\Gamma_{00}^{\beta1})(\Gamma_{00}^{2\beta'}+\Gamma_{00}^{\beta'2})]\leqslant\sum_{\beta,\beta'=0,1,2}[(\Gamma_{00}^{1\beta}+\Gamma_{00}^{\beta1})(\Gamma_{00}^{2\beta'}+\Gamma_{00}^{\beta'2})+(\Gamma_{20}^{1\beta}+\Gamma_{02}^{\beta1})(\Gamma_{00}^{2\beta'}+\Gamma_{00}^{\beta'2})],$$

$$\sum_{\beta,\beta'=0,1,2}(\Gamma_{02}^{1\beta}\Gamma_{10}^{\beta'2}+\Gamma_{00}^{1\beta}\Gamma_{00}^{\beta'2}+\Gamma_{00}^{2\beta}\Gamma_{00}^{\beta'1}+\Gamma_{00}^{\beta1}\Gamma_{01}^{2\beta'})=\sum_{\beta,\beta'=0,1,2}(\Gamma_{00}^{1\beta}\Gamma_{10}^{\beta'2}+\Gamma_{00}^{\beta2}\Gamma_{02}^{1\beta}+\Gamma_{01}^{2\beta}\Gamma_{20}^{\beta'1}+\Gamma_{00}^{2\beta}\Gamma_{00}^{\beta'1}),$$

$$\sum_{\beta,\beta'=0,1,2}[\Gamma_{01}^{1\beta}\Gamma_{10}^{\beta'1}+\Gamma_{02}^{2\beta}\Gamma_{20}^{\beta'2}-4\Gamma_{01}^{2\beta}\Gamma_{02}^{1\beta'}-4\Gamma_{00}^{1\beta}\Gamma_{00}^{2\beta'}+4\Gamma_{00}^{2\beta}\Gamma_{02}^{1\beta'}+4\Gamma_{00}^{1\beta}\Gamma_{01}^{2\beta'}-(\Gamma_{00}^{1\beta}-\Gamma_{00}^{2\beta})(\Gamma_{00}^{\beta'1}-\Gamma_{00}^{\beta'2})]=\sum_{\beta,\beta'=0,1,2}[\Gamma_{02}^{2\beta}\Gamma_{10}^{\beta'1}+\Gamma_{01}^{1\beta}\Gamma_{20}^{\beta'2}-(\Gamma_{00}^{1\beta}-\Gamma_{00}^{2\beta})\times(\Gamma_{01}^{1\beta'}-\Gamma_{02}^{2\beta'})-(\Gamma_{00}^{\beta1}-\Gamma_{00}^{\beta2})(\Gamma_{10}^{\beta'1}-\Gamma_{20}^{\beta'2})]. \tag{38}$$

These are necessary conditions (but not sufficient) for the matrix $\mathcal{M}(p)$ to be nondiagonalizable. This means in turn that it is *sufficient* (but not *necessary*) that one of the relations (38) be violated for the matrix $\mathcal{M}(p)$ to be diagonalizable.

If $\mathcal{M}_{2,1}(p)=0$, the matrix $\mathcal{M}(p)$ is triangular, i.e.,

$$A_1^b+C_2^b=A_2^b=C_1^b=0, \tag{39}$$

which in terms of reaction rates imply

$$\sum_{\beta=0,1,2}(\Gamma_{10}^{2\beta}+\Gamma_{01}^{\beta2}-\Gamma_{00}^{2\beta}-\Gamma_{00}^{\beta2})=\sum_{\beta=0,1,2}(\Gamma_{01}^{2\beta}-\Gamma_{00}^{2\beta})=\sum_{\beta=0,1,2}(\Gamma_{10}^{\beta2}-\Gamma_{00}^{\beta2}). \tag{40}$$

So for example, if $\Gamma_{00}^{2\beta}=\Gamma_{00}^{\beta2}=\Gamma_{01}^{2\beta}=\Gamma_{10}^{\beta2}=\Gamma_{01}^{\beta2}=\Gamma_{10}^{2\beta}$, the relations (40) are fulfilled.

If the matrix $\mathcal{M}$ is diagonal, i.e., $\mathcal{M}_{2,1}(p)=\mathcal{M}_{1,2}(p)=0$, and in addition to (39) and (40), we have

$$B_1^a+D_2^a=B_2^a=D_1^a=0\Rightarrow\sum_{\beta=0,1,2}(\Gamma_{20}^{1\beta}+\Gamma_{02}^{\beta1}-\Gamma_{00}^{1\beta}-\Gamma_{00}^{\beta1})=\sum_{\beta=0,1,2}(\Gamma_{02}^{1\beta}-\Gamma_{00}^{1\beta})=\sum_{\beta=0,1,2}(\Gamma_{20}^{\beta1}-\Gamma_{00}^{\beta1}). \tag{41}$$

As an example, relations (39)–(41) are fulfilled if one has $\Gamma_{00}^{1\beta}=\Gamma_{00}^{\beta1}=\Gamma_{02}^{1\beta}=\Gamma_{20}^{\beta1}=\Gamma_{02}^{\beta1}$ and $\Gamma_{00}^{2\beta}=\Gamma_{00}^{\beta2}=\Gamma_{01}^{2\beta}=\Gamma_{10}^{\beta2}=\Gamma_{10}^{2\beta}=\Gamma_{01}^{\beta2}$.

It follows from this discussion that when the reaction rates, in addition to the solubility constraints (19), also violate conditions (39) and one of the relations (38) which are *sufficient* but not *necessary*, the first case applies. When, in addition to (19), the relations (38), are fulfilled [recall that (38) are *necessary* but not *sufficient* constraints] and the conditions (39) are violated, then the second case applies. When reaction rates satisfy (39)–(40) in addition to the relation (19), then the third case (see below) applies.

Similarly, when reaction-rates satisfy (39)–(41) in addition to the relation (19), then the fourth case (see below) applies.

(3) If $\mathcal{M}_{2,1}(p)=0$, the matrix $\mathcal{M}(p)$ is triangular, thus the eigenvalues of $\mathcal{M}(p)$ are $\mathcal{M}_{1,1}(p)$ and $\mathcal{M}_{2,2}(p)$. We have already discussed the physical implication of this case [see Eq. (40) above], and we have ($\vec{p}\neq 0$),

$$e^{\mathcal{M}(p)t}=e^{\mathcal{M}_{1,1}(p)t}\begin{pmatrix}1 & \mathcal{M}_{1,2}(p)t\\ 0 & 1\end{pmatrix} \tag{42}$$

which leads to ($\vec{p}\neq 0$)

$$\mathcal{S}_0^A(\vec{p},\omega)=\frac{1}{\omega-\mathcal{M}_{1,1}(p)}\left[\langle \hat{n}_p^A\rangle(0)+\frac{\mathcal{M}_{1,2}(p)}{\omega-\mathcal{M}_{1,1}(p)}\langle \hat{n}_p^B\rangle\right],$$
$$\mathcal{S}_0^B(\vec{p},\omega)=\frac{\langle \hat{n}_p^B\rangle(0)}{\omega-\mathcal{M}_{1,1}(p)}. \tag{43}$$

(4) If both $\mathcal{M}_{2,1}(p)=\mathcal{M}_{1,2}(p)=0$, the matrix $\mathcal{M}(p)$ is already diagonal [see (40)], and the form factors read ($\vec{p}\neq 0$),

$$\mathcal{S}_0^A(\vec{p},\omega)=\frac{\langle \hat{n}_p^A\rangle(0)}{\omega-\mathcal{M}_{1,1}(p)},\quad \mathcal{S}_0^B(\vec{p},\omega)=\frac{\langle \hat{n}_p^B\rangle(0)}{\omega-\mathcal{M}_{2,2}(p)}. \tag{44}$$

In the sequel we focus on the case where the matrix $\mathcal{M}(p)$ is diagonal and provide explicit expressions in real space and time for the density. This is equivalent to imposing the six supplementary conditions characterizing V_1 [see (39)–(41)], $B_1^a+D_2^a=B_2^a=D_1^a=A_1^b+C_2^b=A_2^b=C_1^b=0$. Let us first compute the density for the case where translation invariance is broken, i.e., the manifold $V_1\supset V_2$ of dimension $72-(16+6)$. With the above constraints, the densities obey the following equations of motion:

$$\frac{d}{dt}\langle n_m^A\rangle=(A_0^a+C_0^a)d+\langle n_m^A\rangle(t)(A_1^a+C_2^a)d$$
$$+\sum_\alpha [A_2^a\langle n_{m+e^\alpha}^A\rangle(t)+C_1^a\langle n_{m-e^\alpha}^A\rangle(t)], \tag{45}$$

$$\frac{d}{dt}\langle n_m^B\rangle=d(A_0^b+C_0^b)+\langle n_m^B\rangle(t)(B_1^b+D_2^b)d$$
$$+\sum_\alpha [B_2^b\langle n_{m+e^\alpha}^B\rangle(t)+D_1^b\langle n_{m-e^\alpha}^B\rangle(t)]. \tag{46}$$

In order to discuss the solutions of these equations, it is convenient to define the following quantities: $\mu_A\equiv\sqrt{C_1^a/A_2^a}$, $\mu_B\equiv\sqrt{D_1^b/B_2^b}$, $C_A\equiv\sqrt{A_2^aC_1^a}$, and $C_B\equiv\sqrt{B_2^bD_1^b}$. Furthermore, we introduce $B_A\equiv 2(A_1^a+C_2^a)\leqslant 0$, $B_B\equiv 2(B_1^b+D_2^b)\leqslant 0$, $\gamma_A\equiv A_1^a+A_2^a+C_1^a+C_2^a\leqslant 0$, $\gamma_B\equiv B_1^b+B_2^b+D_1^b+D_2^b\leqslant 0$, and $\epsilon_j\equiv\ln\mu_j$, $j=A,B$. A site on the hypercube is denoted by m with d components: $m=(m_1,\dots,m_i,\dots,m_d)$.

Exploiting the well-known properties of Bessel functions [$j\in(A,B)$], we arrive at

$$\langle n_m^j\rangle(t)=\rho_j(\infty)(1-e^{-d|\gamma_j|t})$$
$$+e^{-d|B_j/2|t}\sum_{m=(m_1',\dots,m_d')}\langle n_{m'}^j\rangle(0)$$
$$\times\prod_{i=1,\dots,d}\mu_j^{m_i-m_i'}I_{m_i-m_i'}(2C_jt). \tag{47}$$

The steady densities are, respectively, $\rho_A(\infty)=[(A_0^a+C_0^a)/2|\gamma_A|]$, and $\rho_B(\infty)=[(A_0^b+C_0^b)/2|\gamma_B|]$. Below, we investigate three different cases.

(i) The ($t=0$) initial density (for each species) is given by ρ_j,

$$\langle n_{m'}^j\rangle(t=0)=\rho_j(0)\,\delta_{m,m'}. \tag{48}$$

The asymptotic behavior is then ($m_i\gg 1$ and $u_j=L^2/4|C_j|t$)

$$\langle n_m^j\rangle(t)\sim\rho_j(\infty)+\psi_m^j\frac{e^{-\Theta_jt}}{t^{\phi_j}}, \tag{49}$$

where ψ_m^j are known functions. Further, we assume $|\gamma_j|>0$, since for $|\gamma_j|=0$, Eq. (19) tells us that $\langle n_m^j\rangle(t)=\rho_j(\infty)=\rho_j(0)=cste,\forall t$, where

$$\Theta_j=\min\left[d|\gamma_j|,d\left(\frac{|B_j|}{2}-|C_j|(2+\epsilon^2/2)\right)\right], \tag{50}$$

$$\phi_j=\begin{cases}0 & \text{if}\quad \Theta_j=d|\gamma_j|>0\\ d/2 & \text{if}\quad \Theta_j=d\left(\frac{|B_j|}{2}-|C_j|(2+\epsilon_j^2/2)\right).\end{cases} \tag{51}$$

For $\Theta=0$ and $\gamma\neq 0$, the density decays algebraically, i.e., $\langle n_m^j\rangle(t)\sim\rho_j(\infty)+\psi_m^jt^{-d/2}$.

(ii) Initially, the particles (for each species) are confined in some region of space, i.e.,

$$\langle n_m^j\rangle(t=0)=\begin{cases}n_0^j & \text{if } 0\leqslant m_i\leqslant L/2\\ 0 & \text{otherwise,}\end{cases} \tag{52}$$

$$\langle n_m^{j'\neq j}\rangle(t=0)=\begin{cases}n_0^{j'} & \text{if } L/2<m_i\leqslant L\\ 0 & \text{otherwise.}\end{cases} \tag{53}$$

We have then

$$n_m^j(t)\sim\rho_j(\infty)+e^{-\Theta_jt}\left(\psi_{m,1}^j+\frac{\psi_{m,2}^j}{t^{\phi_j}}\right), \tag{54}$$

where Θ_j has been defined in (50) and for $r=m-m'$, $|r|\gg 1$, $r^2/\mathcal{C}t<\infty$:

$$\phi_j=\begin{cases}0 & \text{if } \Theta_j=d|\gamma_j|>0\\ 1 & \text{if } \Theta_j=d\left(\frac{|B_j|}{2}-C_j(2+\epsilon_j^2/2)\right).\end{cases} \tag{55}$$

If $\Theta=0$ and $\gamma\neq 0$, the density decays as a power law, i.e., $\langle n_m^j\rangle(t)\sim\rho_j(\infty)+[\psi_{m,1}^j+(\psi_{m,2}^j/t)]$.

(iii) The initial distribution of particles is assumed to be nonuniform, i.e.,

$$\langle n_m^j\rangle(t=0)=\begin{cases} n_0^j & \text{if } m=0\\ n_0^j\prod_{i=1,\ldots,d}(1-\delta_{m_i,0})|m_i|^{-\alpha_i} & \text{if } |m|>0.\end{cases} \tag{56}$$

With (56), we arrive at ($m_i\geqslant 1$ and $u_j=L^2/4C_jt$)

$$\langle n_m^j\rangle(t)\sim\rho_j(\infty)+\psi_m^j\,\frac{e^{-\Theta_j t}}{t^{\phi_j}}. \tag{57}$$

When

$$\Theta_j=d\left[\frac{|B_j|}{2}-C_j\left(2+\frac{\epsilon_j^2}{2}\right)\right],$$

decays as (59), with

$$\phi_j=\begin{cases}\sum_i\frac{\alpha_i}{2} & \text{if } 0\leqslant\alpha_i<1\\ \frac{d}{2} & \text{if } \alpha_i\geqslant 1.\end{cases} \tag{58}$$

Again, as $\Theta_j=0$, Eq. (58) holds.

Notice the crossover at $\alpha_i=\alpha=1$, where the density decays as

$$\langle n_m^j\rangle(t)\sim\rho_j(\infty)+\psi_m^j\left(e^{-d\{(|B_j|/2)-|C_j|[2+(\epsilon_j^2/2)]\}t}\left[\frac{\ln(4u_j|C_j|t)}{2\sqrt{4\pi|C_j|t}}\right]^d\right).$$

By contrast, when $\Theta_j=d|\gamma_j|>0$, the density behaves as $\langle n_m^j\rangle(t)\sim\rho_j(\infty)+\psi_m^j e^{-d|\gamma_j|t}$. Here we have restricted our attention to the case where $0<\alpha_i<1$, $\forall i, 0<\alpha_i<1$, while in general we could consider different regimes in the different directions (for example, $0<\alpha_1<1$, $\alpha_1=1$, and $\alpha_1>1$). The corresponding asymptotic behavior follows as above.

To conclude this section let us focus on the manifold V_2, where the density is translationally invariant. On account of (46), we obtain two coupled linear differential equations which are easily integrated. The result is

$$\rho_j(t)=\rho_j(\infty)+[\rho_j(0)-\rho_j(\infty)]e^{-d|\gamma_j|t}. \tag{59}$$

The above solution allows us to solve for the correlation functions on $V_2(d)$. These in turn will be useful to solve perturbatively the problem on the manifold $V_{\text{transl-invar}}(d)$.

Let us now discuss the relationship between our results and the solution of some models solved exactly in $d\geqslant 1$ [11–15] earlier.

In Ref. [11], Clément *et al.*, solved exactly (the fast adsorption rates version) of the Fichthorn, Gulari, and Ziff (FGZ) model [12] introduced to describe the conversion of CO and O to CO_2 on platinum substrates. The model solved in Ref. [11] describes the dynamics of classical stochastic particles of two species (called A and B) *with hard-core constraint*, and so our approach applies to this model. However, as the system solved in Ref. [11] is a *two-states model* (in Ref. [11], the stochastic variable is a ‘‘spin variable’’ $z_j=\pm 1$, $+1$ corresponding to an A particle and -1 to a B particle at site j). In this model there are no vacancies, it is not in the class of (*three-states*) models which we specifically study there. It is, however, possible to recover previous results [11] considering the system [11] in the framework of the *two-states* analog of our approach [7,9]. To do so we relabel Clément *et al.* B particles by vacancies symbolized by 0 (the A particles are symbolized by 1), thus reactions occurring in Ref. [11] are described by the rates: $\Gamma_{11}^{10}=\Gamma_{11}^{01}=\Gamma_{00}^{10}=\Gamma_{00}^{01}=p/4$; $\Gamma_{10}^{01}=\Gamma_{01}^{10}=1/4d$; $\Gamma_{10}^{00}=\Gamma_{10}^{11}=\Gamma_{01}^{11}=\Gamma_{01}^{00}=p/4+1/4d$; $\Gamma_{11}^{11}=\Gamma_{00}^{00}=-p/2$; and $\Gamma_{10}^{10}=\Gamma_{01}^{01}=-(p/2+3/4d)$. For such systems (where $s=1$), the $2s^3=2$ *solubility constraints*, which are the analogs of (19), read [7,9] $\Gamma_{10}^{00}+\Gamma_{00}^{10}+\Gamma_{10}^{01}+\Gamma_{00}^{11}=\Gamma_{11}^{00}+\Gamma_{11}^{01}+\Gamma_{01}^{10}+\Gamma_{01}^{11}$ and $\Gamma_{01}^{00}+\Gamma_{00}^{01}+\Gamma_{01}^{10}+\Gamma_{00}^{11}=\Gamma_{11}^{00}+\Gamma_{11}^{10}+\Gamma_{10}^{01}+\Gamma_{10}^{11}$. These constraints are fulfilled for the previous choice of rates (similar to the choice of [11]) and thus the dynamics of the system is soluble in arbitrary dimensions, i.e., the equations of motion of the correlation functions are closed and obtained as in (14), (17), and (18). As an example, for the density we have the following equation of motion [7]: $(d/dt)\langle\tilde{n}_j\rangle=B_1\Sigma_{\pm\alpha}(\langle\tilde{n}_{j+\alpha}\rangle-\langle\tilde{n}_j\rangle)-p\langle\tilde{n}_{j+\alpha}\rangle$, where $\tilde{n}_j\equiv n_j-\frac{1}{2}$ and $B_1=\Gamma_{10}^{01}+\Gamma_{10}^{11}-\Gamma_{00}^{01}-\Gamma_{00}^{11}=1/2d$ are the same quantities defined in Ref. [7]. Noting that in language of Ref. [11] $\gamma_j\equiv\langle z_j\rangle=2\langle\tilde{n}_j\rangle$, we recover the result of Ref. [11]: $(d/dt)\gamma_j=(1/2d)\triangle\gamma_j-p\gamma_j$, where $\triangle\gamma_j\equiv\Sigma_{\pm\alpha}(\gamma_{j+\alpha}-\gamma_j)$. Similarly we can reproduce the (closed) equations of motion for higher correlation functions. Saturation phenomena as in Ref. [11] should also occur in the class of three states models. However, the analytical treatment would be more complex than in Ref. [11] two-states models.

Let us sketch the strategy which one should follow to treat saturation in the models considered here. For translationally invariant systems one should solve a (linear) differential-difference systems of coupled equations describing equations of motion of correlation functions $\langle n_m^i n_l^j\rangle(t)$, $j\in(A,B)$ paying due attention to the boundary terms $m=l$ and $|m-l|=1$ (see also Sec. V). This system is solved in Fourier space and involves a general 3×3 matrix with nonconstant entries. One should, as it has been done for the density, carefully discuss the properties of this matrix, which is a technical matter. In fact such a study should be carried out for some specific model.

A further two-states model which can be solved exactly in $d\geqslant 1$ is the Voter model (see, e.g., Ref. [16]) described by the reactions rates $\Gamma=\Gamma_{01}^{00}=\Gamma_{10}^{00}=\Gamma_{01}^{11}=\Gamma_{01}^{11}>0$. Since this model fulfills the previous ‘‘two-states’’ *solubility constraints*, it is soluble in arbitrary dimensions. With the *two-states* analog of (14), (17), and (18) we obtain the (closed) equations of motion of the correlation functions. As an example, for the density, we have [16] $d/dt\langle n_j\rangle = 2\Gamma\Sigma_{\alpha}(\langle n_{j+\alpha}\rangle + \langle n_{j-\alpha}\rangle - 2\langle n_j\rangle)$.

Another important model which has been studied rigorously in dimensions $d\geqslant 1$ is the (irreversible) reaction $A+B\to\emptyset+\emptyset$. For this model, Bramson and Lebowitz [14] obtained, rigorously, upper and lower bounds for the long time behavior of densities. However, systems considered in these works, Ref. [14], allow the multiple occupancy of a site by particles of the same species. Later, Belitsky [15] generalized the study of Ref. [14] to the case of hard-core particles reacting according to $A+B\to\emptyset+\emptyset$ [with rates $\Gamma_{12}^{00}=\Gamma_{21}^{00}=\Gamma(=1)$] and $A+\emptyset\leftrightarrow\emptyset+A$; $B+\emptyset\leftrightarrow\emptyset+B$ (with rates $\Gamma_{10}^{01}=\Gamma_{01}^{10}=\Gamma_{20}^{02}=\Gamma_{02}^{20}=1$). He obtained rigorously an upper bound for long-time behavior of the density [for an uncorrelated initial state with equal species densities: $\rho_A(0)=\rho_B(0)\leqslant 1/2$]: $\forall\epsilon>0,\exists T(\epsilon)<\infty$, so that for $t>T(\epsilon)$, $\rho_{t\to\infty}(t)\leqslant t^{-d/4+\epsilon}$, for $d\leqslant 4$ and $\rho_{t\to\infty}(t)\leqslant C^{\star}t^{-1}$, for $d>4$; where $C^{\star}$ is a positive constant. We can now wonder whether such a model can be dealt with in our approach. As the model considered by Belitsky is a *three-states* model, the equations of motion of correlation functions are given by (14), (17), and (18) with $A_2^a=C_1^a=B_2^b=D_1^b=-A_1^a=-B_1^b=-C_2^a=-D_2^b=1$ and $A_0^a=B_1^a=B_2^a=D_1^a=D_2^a=A_0^b=A_1^b=A_2^b=C_0^b=C_1^b=C_2^b=0$. Unfortunately the solubility constraints (19) are not fulfilled for such a model (consider, e.g., the fourth constraint: $A_2^a+B_1^a+A_0^a=\Gamma_{21}^{10}+\Gamma_{21}^{11}+\Gamma_{21}^{12}\Rightarrow\Gamma=0$, but in this model $\Gamma=1$) and the equations of motion give rise to an open hierarchy which cannot be solved.

Results in $d\geqslant 1$ have also been obtained by approximate methods (mean-field theories), and/or by scaling and heuristic arguments (see, e.g., Ref. [1] and references therein). As an illustration of these studies let us consider the work of Toussaint and Wilczek. In Ref. [13], a system of two species A and B reacting according to $A+B\to\emptyset+\emptyset$, in addition to their diffusive motion, is studied numerically and an approximate method for calculating the densities at long times is proposed [for system with equal densities : $\rho_A(0)=\rho_B(0)$]. Approximate results [13] predict $\rho(t)\sim t^{-d/4}$, in agreement with Bramson, Lebowitz [14], and Belitsky [15] rigorous results in the one-dimensional case, but in disagreement in higher dimensions $d>1$. The approach in Ref. [13] is a continuum macroscopic approach and cannot take into account the hard-core constraint of the particles . In addition, it takes into account of fluctuations in an approximate and uncontrolled way. It is therefore difficult to compare their method with the microscopic exact results presented here.

In sum we have seen that the two-states formulation [7] of the method discussed here allows to recover some previous exact results in arbitrary dimensions [11,16] for the stochastic models of hard-core particles. Our approach applied on three-states models (of hard-core particles) is in a sense complementary to the rigorous results of [15] and is useful to describe exactly physical models such as a *three-states* growth model [10].

IV. NONINSTANTANEOUS CORRELATION FUNCTIONS

To our knowledge, the only exact computations of two-time correlation functions in nonequilibrium statistical mechanics that are available are those for single-species models, in particular for one-dimensional models which can be mapped onto free fermion [18,19,22] and other related [21,16] models. These exact results are useful, as starting points for perturbative calculations or for checking numerical computations. This section is devoted to the study of the 50-parametric manifold V_1. We are interested in the density–density correlation functions

$$\langle n_m^i(t)n_l^j(0)\rangle\equiv\langle\tilde{\chi}|n_m^i e^{-Ht}n_l^j|P(0)\rangle = \langle\tilde{\chi}|n_m^i e^{-Ht}|P'(0)\rangle,\quad i,j\in(A,B), \tag{60}$$

where $|P(0)\rangle$ denotes the initial state of the system. From the above we see that the evaluation for the correlation function with respect to the initial state $|P(0)\rangle$ is equivalent to computation of the density of particles of species i at site m for a system in an initial state described by $|P'(0)\rangle \equiv n_l^j|P(0)\rangle$.

We now distinguish the case where correlations are absent in the initial (with broken translation invariance) state from that where they are present.

A. Noninstantaneous two-point correlation functions for uncorrelated initial states

In this section we assume uncorrelated initial states with a random and translationally invariant distribution of particles of type A [density $\rho_A(0)$] and of type B [density $\rho_B(0)$]. Therefore in our notations $|P(0)\rangle$ becomes

$$|P(0)\rangle=\begin{pmatrix}1-\rho_A(0)-\rho_B(0)\\ \rho_A(0)\\ \rho_B(0)\end{pmatrix}^{\otimes L^d}. \tag{61}$$

So that we have

$$\begin{aligned}\langle n_m^A(0)n_l^A(0)\rangle&=\rho_A(0)\delta_{m,l}+\rho_A(0)^2(1-\delta_{m,l}),\\ \langle n_m^B(0)n_l^B(0)\rangle&=\rho_B(0)\delta_{m,l}+\rho_B(0)^2(1-\delta_{m,l}),\\ \langle n_m^A(0)n_l^B(0)\rangle=\langle n_m^B(0)n_l^A(0)\rangle&=\rho_A(0)\rho_B(0)(1-\delta_{m,l}).\end{aligned} \tag{62}$$

We begin this section by computing explicitly the Fourier–Laplace transform of the two-points noninstantaneous correlation functions, i.e., the dynamic form factors measured in the light scattering experiments [17,19]

$$\begin{aligned}\mathcal{S}_1^{ij}(\vec{p},\omega)&\equiv\frac{1}{L^d}\sum_{m'\equiv m-l}\int_0^{\infty}dt\, e^{-i\vec{p}.m'-\omega t}\langle n_m^i(t)n_l^j(0)\rangle\\ &=\frac{1}{L^d}\sum_{m'\equiv m-l}\int_0^{\infty}dt\, e^{-i\vec{p}.m'-\omega t}\langle n^i_{m'=m-l}(t)\,n^j_0(0)\rangle,\\ &\qquad (i,j)\in(A,B).\end{aligned} \tag{63}$$

Using the results of the preceding section and assuming regularity and diagonalizability of $\mathcal{M}(p)$, we find for the dynamic form factors

$$\mathcal{S}_1^{AA}(\vec{p},\omega)=\frac{1}{\lambda_-(p)-\lambda_+(p)}\left[\frac{\lambda_-(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_+(p)}-\frac{\lambda_+(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_-(p)}\right][(\rho_A^2(0)+\rho_A(0))\delta_{p,0}-\rho_A(0)]+\frac{\mathcal{M}_{1,2}(p)[\rho_A(0)\rho_B(0)](\delta_{p,0}-1)}{[\omega-\lambda_+(p)][\omega-\lambda_-(p)]}+\frac{\delta_{p,0}}{\gamma_--\gamma_+}[(\gamma_--\mathcal{M}_{1,1}(0))(A_0^a+C_0^a)d-\mathcal{M}_{1,2}(0)(A_0^b+C_0^b)d]\frac{1}{\omega(\omega-\gamma_+)}-\frac{\delta_{p,0}}{\gamma_--\gamma_+}[(\gamma_--\mathcal{M}_{1,1}(0))(A_0^a+C_0^a)d-\mathcal{M}_{1,2}(0)(A_0^b+C_0^b)d]\frac{1}{\omega(\omega-\gamma_-)}, \quad (64)$$

$$\mathcal{S}_1^{AB}(\vec{p},\omega)=\frac{1}{\lambda_-(p)-\lambda_+(p)}\left[\frac{\lambda_-(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_+(p)}-\frac{\lambda_+(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_-(p)}\right]\rho_A(0)\rho_B(0)(\delta_{p,0}-1)+\frac{\mathcal{M}_{1,2}(p)[(\rho_B^2(0)+\rho_B(0))\delta_{p,0}-\rho_B(0)]}{[\omega-\lambda_+(p)][\omega-\lambda_-(p)]}+\frac{\delta_{p,0}}{\gamma_--\gamma_+}[(\gamma_--\mathcal{M}_{1,1}(0))(A_0^a+C_0^a)d-\mathcal{M}_{1,2}(0)(A_0^b+C_0^b)d]\frac{1}{\omega(\omega-\gamma_+)}-\frac{\delta_{p,0}}{\gamma_--\gamma_+}[(\gamma_--\mathcal{M}_{1,1}(0))(A_0^a+C_0^a)d-\mathcal{M}_{1,2}(0)(A_0^b+C_0^b)d]\frac{1}{\omega(\omega-\gamma_-)} \quad (65)$$

and

$$\mathcal{S}_1^{BB}(\vec{p},\omega)=[(\rho_B^2(0)+\rho_B(0))\delta_{p,0}-\rho_B(0)]\times\left[\frac{\lambda_-(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_-(p)}-\frac{\lambda_+(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_+(p)}\right]-\left[\frac{[\lambda_+(p)-\mathcal{M}_{1,1}(p)][\lambda_-(p)-\mathcal{M}_{1,1}(p)]}{\mathcal{M}_{1,2}(p)(\omega-\lambda_+(p))(\omega-\lambda_-(p))}\right]\times\rho_A(0)\rho_B(0)(\delta_{p,0}-1)+\frac{\delta_{p,0}}{\gamma_--\gamma_+}\times\left[\frac{[\gamma_--\mathcal{M}_{1,1}(0)]}{\mathcal{M}_{1,2}(0)}(A_0^a+C_0^a)d-(A_0^b+C_0^b)d\right]\times\frac{\gamma_+-\mathcal{M}_{1,1}(0)}{\omega(\omega-\gamma_+)}-\frac{\delta_{p,0}}{\gamma_--\gamma_+}\left[\frac{[\gamma_+-\mathcal{M}_{1,1}(0)]}{\mathcal{M}_{1,2}(0)}\times(A_0^a+C_0^a)d-(A_0^b+C_0^b)d\right]\frac{\gamma_--\mathcal{M}_{1,1}(0)}{\omega(\omega-\gamma_-)}, \quad (66)$$

$$\mathcal{S}_1^{BA}(\vec{p},\omega)=\rho_A(0)\rho_B(0)(\delta_{p,0}-1)\left[\frac{\lambda_-(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_-(p)}-\frac{\lambda_+(p)-\mathcal{M}_{1,1}(p)}{\omega-\lambda_+(p)}\right]-\left[\frac{[\lambda_+(p)-\mathcal{M}_{1,1}(p)][\lambda_-(p)-\mathcal{M}_{1,1}(p)]}{\mathcal{M}_{1,2}(p)[\omega-\lambda_+(p)][\omega-\lambda_-(p)]}\right]\times[(\rho_A^2(0)+\rho_A(0))\delta_{p,0}-\rho_A(0)]+\frac{\delta_{p,0}}{\gamma_--\gamma_+}\left[\frac{[\gamma_--\mathcal{M}_{1,1}(0)]}{\mathcal{M}_{1,2}(0)}(A_0^a+C_0^a)d-(A_0^b+C_0^b)d\right]\frac{\gamma_+-\mathcal{M}_{1,1}(0)}{\omega(\omega-\gamma_+)}-\frac{\delta_{p,0}}{\gamma_--\gamma_+}\times\left[\frac{[\gamma_+-\mathcal{M}_{1,1}(0)]}{\mathcal{M}_{1,2}(0)}(A_0^a+C_0^a)d-(A_0^b+C_0^b)d\right]\frac{\gamma_--\mathcal{M}_{1,1}(0)}{\omega(\omega-\gamma_-)}. \quad (67)$$

Again the poles of the dynamic form factors give the relaxational eigenvalues. As in the preceding section, we could also compute the correlation functions in the case where $\mathcal{M}(p)$ is nondiagonalizable, triangular or already diagonal, but for brevity's sake we prefer to focus here on the noninstantaneous correlation functions on the 50-parameters manifold V_1.

With the help of Eqs. (45)–(47), we obtain the noninstantaneous two-point correlation functions on the manifold V_1 as

$$\langle n_m^A(t)n_l^A(0)\rangle=\rho_A(\infty)(1-e^{-d|\gamma_A|t})+\rho_A^2(0)e^{-d|\gamma_A|t}+[\rho_A(0)-\rho_A^2(0)]\times\prod_{\alpha=1\ldots d}\mu_A^{m_\alpha-l_\alpha}e^{-(|B_A|t/2)}I_{m_\alpha-l_\alpha}(2C_At), \quad (68)$$

$$\langle n_m^A(t)n_l^B(0)\rangle=\rho_A(\infty)(1-e^{-d|\gamma_A|t})+\rho_A(0)\rho_B(0)\times\left[e^{-d|\gamma_A|t}-\prod_{\alpha=1,\ldots,d}\mu_A^{m_\alpha-l_\alpha}e^{-(|B_A|t/2)}\times I_{m_\alpha-l_\alpha}(2C_At)\right], \quad (69)$$

$$\langle n_m^B(t)n_l^B(0)\rangle=\rho_B(\infty)(1-e^{-d|\gamma_B|t})+\rho_B^2(0)e^{-d|\gamma_B|t}+[\rho_B(0)-\rho_B^2(0)]\prod_{\alpha=1,\ldots,d}\mu_B^{m_\alpha-l_\alpha}e^{-(|B_B|t/2)}I_{m_\alpha-l_\alpha}(2C_Bt), \quad (70)$$

$$\langle n_m^B(t) n_l^A(0)\rangle = \rho_B(\infty)(1-e^{-d|\gamma_B|t}) + \rho_A(0)\rho_B(0) \times\left[e^{-d|\gamma_B|t} - \prod_{\alpha=1,\dots,d} \mu_B^{m_\alpha - l_\alpha} e^{-(|B_B|t/2)} \times I_{m_\alpha - l_\alpha}(2C_B t)\right] \quad (71)$$

with the notations, $\vec{r}\equiv\Sigma_\alpha r_\alpha e_\alpha$, $\vec{\epsilon_i}\equiv\epsilon_i\Sigma_\alpha e_\alpha$, and $r_\alpha = m_\alpha - l_\alpha = \sigma L$. We are interested in the asymptotic behavior $[|C_j|t\gg 1, j\in(A,B)$ and $u_j = L^2/4|C_j|t<\infty]$ of the above correlation functions in two regimes: (i) when $|m-l|\equiv|r|\equiv(\Sigma_{\alpha=1,\dots,d} r_\alpha^2)^{1/2}\sim L\gg 1$, in this case $\sigma = r/L = \mathcal{O}(1)$; (ii) when $|m-l|\equiv|r|\ll 1$, in this case $\sigma = r/L = \mathcal{O}(1/L)$.

It is worth noting that the autocorrelation functions (where $|m-l|=0$) are obtained in the second regimes (ii). With the above, we finally arrive at

$$\langle n_m^A(t) n_l^A(0)\rangle = \rho_A(\infty)(1-e^{-d|\gamma_A|t}) + \rho_A^2(0)e^{-d|\gamma_A|t} + \exp\left(2\,d\sigma^2 u_A - \frac{(\vec{r}-\vec{\epsilon}_A|C_A|t)^2}{2|C_A|t}\right) e^{-[(|B_A|/2)-|C_A|(2+\epsilon_A^2/2)]t} \times\left[\frac{[\rho_A(0)-\rho_A^2(0)]e^{-d\sigma^2 u_A}}{(4\pi|C_A|t)^{d/2}} + \mathcal{O}(1/t^d)\right], \quad (72)$$

$$\langle n_m^A(t) n_l^B(0)\rangle = \rho_A(\infty)(1-e^{-d|\gamma_A|t}) + \rho_A(0)\rho_B(0)e^{-d|\gamma_A|t} + \exp\left(2\,d\sigma^2 u_A - \frac{(\vec{r}-\vec{\epsilon}_A|C_A|t)^2}{2|C_A|t}\right) \times e^{-d[(|B_A|/2)-|C_A|(2+\epsilon_A^2/2)]t}\rho_A(0)\rho_B(0) \times\left[1-\frac{e^{-d\sigma^2 u_A}}{(4\pi|C_A|t)^{d/2}} + \mathcal{O}(1/t^d)\right], \quad (73)$$

$$\langle n_m^B(t) n_l^B(0)\rangle = \rho_B(\infty)(1-e^{-d|\gamma_B|t}) + \rho_B^2(0)e^{-d|\gamma_B|t} + \exp\left(2d\sigma^2 u_B - \frac{(\vec{r}-\vec{\epsilon}_B|C_B|t)^2}{2|C_B|t}\right) \times e^{-d[(|B_B|/2)-|C_B|(2+\epsilon_B^2/2)]t} \times\left[\frac{[\rho_B(0)-\rho_B^2(0)]e^{-d\sigma^2 u_B}}{(4\pi|C_A|t)^{d/2}} + \mathcal{O}(1/t^d)\right], \quad (74)$$

$$\langle n_m^B(t) n_l^A(0)\rangle = \rho_B(\infty)(1-e^{-d|\gamma_B|t}) + \rho_A(0)\rho_B(0)e^{-d|\gamma_B|t} + \exp\left(2d\sigma^2 u_B - \frac{(\vec{r}-\vec{\epsilon}_B|C_B|t)^2}{2|C_B|t}\right) \times e^{-d[(|B_B|/2)-|C_B|(2+\epsilon_B^2/2)]t}\rho_A(0)\rho_B(0) \times\left[1-\frac{e^{-d\sigma^2 u_B}}{(4\pi|C_B|t)^{d/2}} + \mathcal{O}(1/t^d)\right]. \quad (75)$$

In the regime $|B_i| = |C_i|(2+\epsilon_i^2/2d)$, $i\in(A,B)$, the two-point correlation functions decay as

$$\langle n_m^i(t) n_l^j(0)\rangle \sim \frac{\exp\left(2\,d\sigma^2 u_i - \frac{(\vec{r}-\vec{\epsilon}_i|C_i|t)^2}{2|C_i|t}\right)}{|C_i|t^{d/2}} \qquad (i,j)\in(A,B). \quad (76)$$

Note the nontrivial dependence on dimensionality and the drift term in

$$\exp\left(2\,d\sigma^2 u_i - \frac{(\vec{r}-\vec{\epsilon}_i|C_i|t)^2}{2|C_i|t}\right), \quad (\epsilon_i\neq 0).$$

We remark that this is consistent with the result obtained in one dimension for free fermions [19]. If $\epsilon_i=0$, than there is no drift:

$$\exp\left(2\,d\sigma^2 u_i - \frac{(\vec{r}-\vec{\epsilon}_i|C_i|t)^2}{2|C_i|t}\right) = 1.$$

B. Noninstantaneous two-point correlation functions on V_1: Correlated initial states

Let us consider correlated initial states described by a distribution having the following properties: (i) when $\mathrm{dist}(m-l)>0$,

$$\langle n_m^i(0) n_l^j(0)\rangle = \mathcal{K}_{ij}\prod_{\alpha=1,\dots,d}(1-\delta_{r_\alpha,0})|r_\alpha|^{-\Delta_{ij}^\alpha},$$

$$\Delta_{ij}>0, \quad \mathcal{K}_{ij}>0, \quad r_\alpha\equiv|m_\alpha - l_\alpha|, \quad (i,j)\in(A,B); \quad (77)$$

(ii) when $m=l$,

$$\langle n_m^i(0) n_l^j(0)\rangle = \langle n_m^i(0)\rangle\delta_{i,j} = \langle n_l^j(0)\rangle\delta_{i,j} = \rho_i(0)\,\delta_{i,j}. \quad (78)$$

The initial distribution in this section has been chosen in a special form, namely in such a way that computations can be carried out explicitly to the end. There is *a priori* no physical justification for such a choice, which has already been considered in Ref. [21] for a single-species reaction-diffusion system. However and most importantly, our goal here is to investigate the dependence of the asymptotics on the initial correlations.

We remark that in one dimension the initial state (77) and (78) is translationally invariant,

$$\langle n_m^i(0)n_l^j(0)\rangle = \langle n^i_{|r|=|m-l|}(0)n_0^j(0)\rangle = \mathcal{K}_{ij}(1-\delta_{|r|,0})|r|^{-\Delta_{ij}} + \rho_i(0)\,\delta_{i,j}\delta_{|r|,0} \tag{79}$$

while in higher dimensions $[d\geqslant 2$, see (77)] translational invariance is broken. This state of affairs has lead us to distinguish in the discussions the one-dimensional case from its higher dimensional counterparts.

(i) We begin with one dimension $(d=1)$, where $r=r_\alpha \equiv m-l$. Because of the translational invariance in the initial state, we expect the noninstantaneous correlation function to depend on $r=m-l$, indeed,

$$\langle n_m^A(t)n_l^A(0)\rangle = \langle n_r^A(t)n_0^A(0)\rangle = \rho_A(\infty)(1-e^{-|\gamma_A|t}) + \rho_A(0)e^{-(|B_A|t/2)}\mu_A^r I_r(2C_At) + \mathcal{K}_{AA}\sum_{r'\neq 0}\mu_A^{r-r'}|r'|^{-\Delta_{AA}}e^{-(|B_A|t/2)} \times I_{r-r'}(2C_At), \tag{80}$$

$$\langle n_m^A(t)n_l^B(0)\rangle = \rho_A(\infty)(1-e^{-|\gamma_A|t}) + \mathcal{K}_{AB}\sum_{r'\neq 0}\mu_A^{r-r'}|r'|^{-\Delta_{AB}}e^{-(|B_A|t/2)} \times I_{r-r'}(2C_At), \tag{81}$$

$$\langle n_m^B(t)n_l^B(0)\rangle = \langle n_r^B(t)n_0^B(0)\rangle = \rho_B(\infty)(1-e^{-|\gamma_B|t}) + \rho_B(0)e^{-(|B_B|t/2)}\mu_B^r I_r(2C_Bt) + \mathcal{K}_{BB}\sum_{r'\neq 0}\mu_B^{r-r'}|r'|^{-\Delta_{BB}}e^{-(|B_B|t/2)} \times I_{r-r'}(2C_Bt), \tag{82}$$

$$\langle n_m^B(t)n_l^A(0)\rangle = \rho_B(\infty)(1-e^{-|\gamma_B|t}) + \mathcal{K}_{BA}\sum_{r'\neq 0}\mu_B^{r-r'}|r'|^{-\Delta_{BA}}e^{-(|B_B|t/2)} \times I_{r-r'}(2C_Bt). \tag{83}$$

Notice that when $\mu_{A,B}\neq 0$, then $\langle n_r^{A,B}(t)n_0^{A,B}(0)\rangle \neq \langle n_{-r}^{A,B}(t)n_0^{A,B}(0)\rangle$ because of the drift which is due to an asymmetric Markov generator H. Such a behavior has been observed in single-species one-dimensional free-fermionic models [19,18].

(ii) In higher dimensions $(d\geqslant 2)$, the initial state is no longer translationally invariant and with (77), (78), and (47), we find

$$\langle n_m^A(t)n_l^A(0)\rangle = \rho_A(\infty)(1-e^{-d|\gamma_A|t}) + \rho_A(0)e^{-(d|B_A|t/2)} \times \prod_{\alpha=1,\dots,d}\mu_A^{m_\alpha-m'_\alpha}I_{m_\alpha-l_\alpha}(2C_At) + \mathcal{K}_{AA}\sum_{(m'_1\neq l_1,\dots,m'_d\neq l_d)} \times \prod_{\alpha=1,\dots,d}\mu_A^{m_\alpha-m'_\alpha}|m'_\alpha-l_\alpha|^{-\Delta^\alpha_{AA}}e^{-(|B_A|t/2)} \times I_{m_\alpha-m'_\alpha}(2C_At), \tag{84}$$

$$\langle n_m^A(t)n_l^B(0)\rangle = \rho_A(\infty)(1-e^{-d|\gamma_A|t}) + \mathcal{K}_{AB} \times \sum_{(m'_1\neq l_1,\dots,m'_d\neq l_d)}\prod_{\alpha=1,\dots,d}\mu_A^{m_\alpha-m'_\alpha} \times |m'_\alpha-l_\alpha|^{-\Delta^\alpha_{AB}}e^{-(|B_A|t/2)}I_{m_\alpha-m'_\alpha}(2C_At), \tag{85}$$

$$\langle n_m^B(t)n_l^B(0)\rangle = \rho_B(\infty)(1-e^{-d|\gamma_B|t}) + \rho_B(0) \times e^{-(d|B_B|t/2)}\prod_{\alpha=1,\dots,d}\mu_B^{m_\alpha-m'_\alpha} \times I_{m_\alpha-m'_\alpha}(2C_Bt) + \mathcal{K}_{BB} \times \sum_{(m'_1\neq l_1,\dots,m'_d\neq l_d)}\prod_{\alpha=1,\dots,d}\mu_B^{m_\alpha-m'_\alpha} \times |m'_\alpha-l_\alpha|^{-\Delta^\alpha_{BB}}e^{-(|B_B|t/2)})I_{m_\alpha-m'_\alpha}(2C_Bt), \tag{86}$$

$$\langle n_m^B(t)n_l^A(0)\rangle = \rho_B(\infty)(1-e^{-d|\gamma_B|t}) + \mathcal{K}_{BA} \times \sum_{(m'_1\neq l_1,\dots,m'_d\neq l_d)}\prod_{\alpha=1,\dots,d}\mu_B^{m_\alpha-m'_\alpha} \times |m'_\alpha-l_\alpha|^{-\Delta^\alpha_{BA}}e^{-(|B_B|t/2)}I_{m_\alpha-m'_\alpha}(2C_Bt). \tag{87}$$

We see that in higher dimensions, because of the broken symmetry of the initial state, the noninstantaneous correlation functions no longer depends on $r_\alpha=m_\alpha-l_\alpha$. We can study the asymptotic behavior of these noninstantaneous correlation functions in an unified way (including both $d=1$ and $d\geqslant 2$), namely for $r_\alpha=m_\alpha-l_\alpha$, with $|r_\alpha|=|m_\alpha-l_\alpha|\sim|m_\alpha| \gg 1$, with $r_\alpha=\sigma_\alpha L$ and $u_i=L^2/4|C_i|t<\infty$, in the regime where $|C_i|t,r\gg 1$, $(i,j)\in(A,B)$, the correlation functions are given by

$$\langle n_m^i(t)n_l^j(0)\rangle=\rho_i(\infty)(1-e^{-d|\gamma_i|t})+\exp\left[2\sum_{\alpha}\sigma_\alpha^2 u_i-\left(\frac{\vec{r}-\vec{\epsilon}_i|C_i|t}{2|C_i|t}\right)^2\right]e^{-d[(|B_i|/2)-|C_i|(2+\epsilon_i^2/2)]t}\times\left(\frac{\rho_i(0)e^{-\sum_\alpha \sigma_\alpha^2 u_i}\delta_{i,j}}{(4\pi|C_i|t)^{d/2}}+\mathcal{K}_{ij}\prod_{\alpha=1,\dots,d}\left[\frac{e^{-\sigma_\alpha^2 u_i}}{1-\Delta_{ij}^\alpha}\sqrt{\frac{u_i\sigma_\alpha^2}{\pi}}\frac{1}{4u_i|C_i|\sigma_\alpha^2 t^{\Delta_{ij}^\alpha/2}}\right]+\mathcal{O}(t^{-2d})\right),\quad 0<\Delta_{ij}^\alpha<1, \tag{88}$$

where we use $\vec{r}\equiv\Sigma_\alpha r_\alpha e_\alpha$ and $\vec{\epsilon}_i\equiv\epsilon_i\Sigma_\alpha e_\alpha$. Moreover,

$$\langle n_m^i(t)n_l^j(0)\rangle=\rho_i(\infty)(1-e^{-d|\gamma_i|t})+\exp\left[2\sum_{\alpha}\sigma_\alpha^2 u_i-\left(\frac{\vec{r}-\vec{\epsilon}_i|C_i|t}{2|C_i|t}\right)^2\right]\times e^{-d[(|B_i|/2)-|C_i|(2+\epsilon_i^2/2)]t}\times\frac{1}{(4\pi|C_i|t)^{d/2}}\left(\rho_i(0)e^{-\sum_\alpha \sigma_\alpha^2 u_i}\delta_{i,j}+\mathcal{K}_{ij}\prod_{\alpha=1,\dots,d}\zeta(\Delta_{ij}^\alpha)+\mathcal{O}(t^{-2d})\right),\quad \Delta_{ij}^\alpha>1. \tag{89}$$

When $\Delta_{ij}^\alpha=1$, a crossover takes place and logarithmic corrections arise, namely,

$$\langle n_m^i(t)n_l^j(0)\rangle=\rho_i(\infty)(1-e^{-d|\gamma_i|t})+\exp\left[2\sum_{\alpha}\sigma_\alpha^2 u_i-\left(\frac{\vec{r}-\vec{\epsilon}_i|C_i|t}{2|C_i|t}\right)^2\right]\times e^{-d[(|B_i|/2)-|C_i|(2+\epsilon_i^2/2)]t}\frac{1}{(4\pi|C_i|t)^{d/2}}\times\left(\rho_i(0)e^{-\sum_\alpha \sigma_\alpha^2 u_i}\delta_{i,j}+\mathcal{K}_{ij}\times\prod_{\alpha=1,\dots,d}\ln(4u_i\sigma_\alpha|C_i|t)+\mathcal{O}(t^{-2d})\right),\quad \Delta_{ij}^\alpha=1. \tag{90}$$

Therefore, when the spatial correlations are important, i.e., $\Delta_{ij}^\alpha<1$, correlation functions (at $[(|B_i|/2)-|C_i|(2+\epsilon_i^2/2d)]=0$) decay as

$$\langle n_m^i(t)n_l^j(0)\rangle\sim\frac{\exp\left[2\sum_{\alpha}\sigma_\alpha^2 u_i-\left(\frac{\vec{r}-\vec{\epsilon}_i|C_i|t}{2|C_i|t}\right)^2\right]}{(4\pi|C_i|t)^{\sum_\alpha \Delta_{ij}^\alpha/2}},\quad \Delta_{ij}^\alpha<1. \tag{91}$$

On the contrary, weak spatial initial correlations do not affect the long time behavior of correlation functions, since

$$\langle n_m^i(t)n_l^j(0)\rangle\sim\frac{\exp\left[2\sum_{\alpha}\sigma_\alpha^2 u_i-\left(\frac{\vec{r}-\vec{\epsilon}_i|C_i|t}{2|C_i|t}\right)^2\right]}{(4\pi|C_i|t)^{d/2}},\quad \Delta_{ij}^\alpha>1. \tag{92}$$

The marginal case $\Delta_{ij}^\alpha=1$ has logarithmic corrections

$$\langle n_m^i(t)n_l^j(0)\rangle\sim\frac{\exp\left[2\sum_{\alpha}\sigma_\alpha^2 u_i-\left(\frac{\vec{r}-\vec{\epsilon}_i|C_i|t}{2|C_i|t}\right)^2\right](\ln 4|C_i|t)^d}{(4\pi|C_i|t)^{d/2}},\quad \Delta_{ij}^\alpha=1. \tag{93}$$

Notice the drift which occurs for $\epsilon\neq0$ and the effect of dimensionality. The fact that initially the state is translationally invariant ($d=1$) gives rise to the asymptotic behavior ($t^{-d/2}$) as for the nontranslationally invariant system in higher dimensions ($d\geqslant2$).

V. INSTANTANEOUS TWO-POINT CORRELATION FUNCTIONS ON THE MANIFOLD V_2

We now pass to the computation of the two-point correlation function on the translation–invariant manifold $V_2(d)$ (21). From (17) and (18), the evolution equations of correlation functions follow. We shall discuss both cases, when the initial state is correlated and when it is uncorrelated. We shall evaluate

$$\begin{aligned}\mathcal{G}_{|r|=|n-m|}^{AA}(t)&\equiv\langle n_n^A n_m^A\rangle(t)\equiv\langle n_{|m-n|}^A n_0^A\rangle(t)\equiv\mathcal{G}_r^{AA}(t),\\ \mathcal{G}_{|r|=|n-m|}^{BB}(t)&\equiv\langle n_n^B n_m^B\rangle(t)\equiv\langle n_{|m-n|}^B n_0^B\rangle(t)\equiv\mathcal{G}_r^{BB}(t),\\ \mathcal{G}_{|r|=|n-m|}^{AB}(t)&\equiv\langle n_n^A n_m^B\rangle(t)\equiv\langle n_n^B n_m^A\rangle(t)\equiv\langle n_{|m-n|}^A n_0^B\rangle(t)\\ &\equiv\langle n_{|m-n|}^B n_0^A\rangle(t)\equiv\mathcal{G}_r^{AB}(t)\equiv\mathcal{G}_r^{BA}(t)\end{aligned} \tag{94}$$

with the boundary conditions at $r=0$ [for the densities see (57)]:

$$\mathcal{G}_{r=0}^{AA}(t)\equiv\rho_A(t),\quad \mathcal{G}_{r=0}^{BB}(t)\equiv\rho_B(t),\quad \mathcal{G}_{r=0}^{AB}(t)\equiv0. \tag{95}$$

Notice that the point $|r|=1$ must be dealt with care. Further, it will be convenient to distinguish the one-dimensional problem from that in $d\geqslant 2$. In this section, in addition to the definition of the Appendix A, we also introduce some specific notations and abbreviations [see Appendix B, Eqs. (B1)–(B3)]. Let us start the discussion with one spatial dimension.

A. One-dimensional instantaneous two-point correlation functions on $V_2(d=1)$

On account of the above remarks and (17), the equations of motion for the correlation functions read ($|r|>1$)

$$\frac{d}{dt}\mathcal{G}_r^{AA}(t)=B_A\mathcal{G}_r^{AA}(t)+\mathcal{C}_A[\mathcal{G}_{r+1}^{AA}(t)+\mathcal{G}_{r-1}^{AA}(t)]+\mathcal{A}_A\rho_A(t) \tag{96}$$

while for $|r|=1$, we have (18),

$$\frac{d}{dt}\mathcal{G}_1^{AA}(t)=(G_1^a+B_A/2)\mathcal{G}_1^{AA}(t)+\mathcal{C}_A\mathcal{G}_2(t)+E_0^a+(F_1^a+F_2^a+\mathcal{A}_A/2)\rho_A(t)+(F_3^a+F_4^a)\rho_B(t). \tag{97}$$

For $r=0$, we recover

$$\frac{d}{dt}\mathcal{G}_0^{AA}(t)=\frac{d}{dt}\rho_A(t)=\frac{\mathcal{A}_A}{2}+\left(\frac{B_A+2\mathcal{C}_A}{2}\right)\rho_A(t). \tag{98}$$

The solution of the above set of coupled differential equations (96)–(98) can be expressed in terms of the modified Bessel functions $I_\nu(z)$ (see Appendix C), and for $|r|\equiv|n-m|>0$ we have

$$\begin{aligned}\mathcal{G}_r^{AA}(t)-[\rho_A(t)]^2=&-[\rho_A(0)e^{-|\gamma_A|t}]^2+\left(\rho_A(0)+\frac{\mathcal{D}_{2,0}^A+\mathcal{D}_{2,1}^A+\mathcal{D}_{2,2}^A}{\mathcal{C}_A}\right)e^{-|B_A|t}I_r(2\mathcal{C}_At)+\sum_{r'\neq 0}\mathcal{G}_{r'}^{AA}(0)e^{-|B_A|t}I_{r-r'}(2\mathcal{C}_At)\\&+\left(\mathcal{D}_{1,0}^A+\frac{\mathcal{D}_{2,0}^A|B_A|}{\mathcal{C}_A}\right)\int_0^t d\tau\, e^{-|B_A|\tau}I_r(2\mathcal{C}_A\tau)+\left(\mathcal{D}_{1,1}^A+\frac{\mathcal{D}_{2,1}^A(|B_A|-|\gamma_A|)}{\mathcal{C}_A}\right)e^{-|\gamma_A|t}\\&\times\int_0^t d\tau\, e^{-(|B|-|\gamma_A|)\tau}I_r(2\mathcal{C}_A\tau)+\left(\frac{\mathcal{D}_{2,2}^A(|B_A|-|\gamma_B|)}{\mathcal{C}_A}\right)e^{-|\gamma_B|t}\int_0^t d\tau\, e^{-(|B_A|-|\gamma_B|)\tau}I_r(2\mathcal{C}_A\tau)\\&-\int_0^t dt'\, e^{-|B_A|(t-t')}\mathcal{G}_1^{AA}(t')\left[\frac{(G_1^a-B_A/2)}{\mathcal{C}_A}\frac{\partial}{\partial t'}I_r[2\mathcal{C}_A(t-t')]+2\mathcal{C}_AI_r[2\mathcal{C}_A(t-t')]\right].\end{aligned} \tag{99}$$

Similarly, we find

$$\begin{aligned}\mathcal{G}_r^{BB}(t)-(\rho_B(t))^2=&-[\rho_B(0)e^{-|\gamma_B|t}]^2+\left(\rho_B(0)+\frac{\mathcal{D}_{2,0}^B+\mathcal{D}_{2,1}^B+\mathcal{D}_{2,2}^B}{\mathcal{C}_B}\right)e^{-|B_B|t}I_r(2\mathcal{C}_Bt)+\sum_{r'\neq 0}\mathcal{G}_{r'}^{BB}(0)e^{-|B_A|t}I_{r-r'}(2\mathcal{C}_Bt)\\&+\left(\mathcal{D}_{1,0}^B+\frac{\mathcal{D}_{2,0}^B|B_B|}{\mathcal{C}_B}\right)\int_0^t d\tau\, e^{-|B_B|\tau}I_r(2\mathcal{C}_B\tau)+\left(\mathcal{D}_{1,1}^B+\frac{\mathcal{D}_{2,1}^B(|B_B|-|\gamma_B|)}{\mathcal{C}_B}\right)e^{-|\gamma_B|t}\\&\times\int_0^t d\tau\, e^{-(|B_B|-|\gamma_B|)\tau}I_r(2\mathcal{C}_B\tau)+\left(\frac{\mathcal{D}_{2,2}^B(|B_B|-|\gamma_A|)}{\mathcal{C}_B}\right)e^{-|\gamma_A|t}\\&\times\int_0^t d\tau\, e^{-(|B_B|-|\gamma_A|)\tau}I_r(2\mathcal{C}_B\tau)-\int_0^t dt'\, e^{-|B_B|(t-t')}\mathcal{G}_1^{BB}(t')\\&\times\left[\frac{(G_2^b-B_B/2)}{\mathcal{C}_B}\frac{\partial}{\partial t'}I_r[2\mathcal{C}_B(t-t')]+2\mathcal{C}_BI_r[2\mathcal{C}_B(t-t'))\right],\end{aligned} \tag{100}$$

and we also obtain

$$\begin{aligned}\mathcal{G}_r^{AB}(t)-[\rho_A(t)\rho_B(t)]=&[\rho_A(\infty)\rho_B(0)+\rho_A(0)\rho_B(\infty)-\rho_A(\infty)\rho_B(\infty)]e^{-|\gamma_A+\gamma_B|t}\\&+\left(\frac{\mathcal{D}_{2,0}^{AB}+\mathcal{D}_{2,1}^{AB}+\mathcal{D}_{2,2}^{AB}}{\mathcal{C}_{AB}}\right)e^{-|B_{AB}|t}I_r(2\mathcal{C}_{AB}t)+\sum_{r'\neq 0}\mathcal{G}_{r'}^{AB}(0)e^{-|B_{AB}|t}I_{r-r'}(2\mathcal{C}_{AB}t)\\&+\left(\mathcal{D}_{1,0}^{AB}+\frac{\mathcal{D}_{2,0}^{AB}|B_{AB}|}{\mathcal{C}_{AB}}\right)\int_0^t d\tau\, e^{-|B_{AB}|\tau}I_r(2\mathcal{C}_{AB}\tau)\\&+\left(\mathcal{D}_{1,1}^{A}+\frac{\mathcal{D}_{2,1}^{AB}(|B_{AB}|-|\gamma_A|)}{\mathcal{C}_{AB}}\right)e^{-|\gamma_A|t}\int_0^t d\tau\, e^{-(|B_{AB}|-|\gamma_A|)\tau}I_r(2\mathcal{C}_{AB}\tau)\end{aligned} \tag{101}$$

$$+\left(\mathcal{D}^{AB}_{1,2}+\frac{\mathcal{D}^{AB}_{2,2}(|B_{AB}|-|\gamma_B|)}{\mathcal{C}_{AB}}\right)e^{-|\gamma_B|t}\int_0^t d\tau\, e^{-(|B_{AB}|-|\gamma_B|)\tau}I_r(2\mathcal{C}_{AB}\tau)$$

$$-\int_0^t dt'\, e^{-|B_{AB}|(t-t')}\mathcal{G}^{AB}_1(t')\left(\frac{H_1^{ab}+H_2^{ab}-A_1^a-D_2^b}{\mathcal{C}_{AB}}\right)\frac{\partial}{\partial t'}I_r[2\mathcal{C}_{AB}(t-t')]$$

$$-2\mathcal{C}_{AB}\int_0^t dt'\, e^{-|B_{AB}|(t-t')}\mathcal{G}^{AB}_1(t')I_r[2\mathcal{C}_{AB}(t-t')].$$

To study the asymptotic behavior of these expressions, we shall distinguish two regimes: (i) long time, i.e., $|\mathcal{C}_j|t\gg 1$ and large distances , i.e., $r\sim L\gg 1$ with the ratios $r^2/|\mathcal{C}_j|t<\infty$ and $u_l\equiv L^2/4|\mathcal{C}_l|t<\infty$ hold finite, and (ii) long time , i.e., $|\mathcal{C}_j|t\gg 1$ and finite distances, $r\ll L\to\infty$ with $r^2/|\mathcal{C}_j|t\ll 1$ and $u_l\equiv L^2/4|\mathcal{C}_l|t<\infty$.

In order to investigate the effect of initial correlations on the dynamics, we consider

$$\mathcal{G}^l_{|r|>0}(0)=\rho_i(0)\rho_j(0)[1-|\kappa_l|(\text{sign}(\mathcal{C}_l))^r r^{-\nu_l}],\qquad \nu_l\geqslant 0, \tag{102}$$

where $[l\in(AA,BB,AB),\ \ i,j\in A,B]$.

Such a choice has been made for the one-dimensional single-species symmetric $A+A\leftrightarrow\varnothing+\varnothing$ process [21]. In this section, we want to proceed with a systematic study of instantaneous correlations, on the manifold $V_2(d=1)$, with the choice (102) for the initial state.

Below we shall use the incomplete gamma function, $\Gamma(\nu,u)\equiv\int_u^\infty dx\, e^{-x}x^{\nu-1}$, as well as the Riemann zeta functions $\zeta(\nu)=\Sigma_{k\geqslant 1}k^{-\nu}$, $\nu>1$. For notational simplicity, we write ν instead of ν_l. The results for the asymptotics are summarized as follows:

(1) For $|B_l|>2|\mathcal{C}_l|$, the decay of correlations is exponential. With the definition

$$\varphi_{jj}\equiv\min(|\gamma_j|,|B_j|-2|\mathcal{C}_j|), \tag{103}$$

we have

(1a) for $\varphi_{jj}=|\gamma_A|\neq|B_j|-2|\mathcal{C}_j|$,

$$\mathcal{G}^{jj}_r(t)-\mathcal{G}^{jj}_r(\infty)\sim e^{-|\gamma_j|t}; \tag{104}$$

(1b) for $\varphi_{jj}=|B_j|-2|\mathcal{C}_j|>0$ and $\mathcal{G}_{|r|>0}(0)\neq 0$,

$$\mathcal{G}^{jj}_r(t)-\mathcal{G}^{jj}_r(\infty)\sim\rho_j(0)^2e^{-(|B_j|-2|\mathcal{C}_j|)t}. \tag{105}$$

Note that in the case where $\mathcal{G}_r(0)=0,\forall r$, we have

$$\mathcal{G}^{jj}_r(t)-\mathcal{G}^{jj}_r(\infty)\sim\begin{cases}\dfrac{Q_1(u_j,\sigma)e^{-(|B_j|-2|\mathcal{C}_j|)t}}{\sqrt{4\pi|\mathcal{C}_j|t}} & \text{if } r\ll L\\[2ex] \dfrac{Q_1(u_j,\sigma)e^{-\sigma^2u_j-(|B_j|-2|\mathcal{C}_j|)t}}{\sqrt{4\pi|\mathcal{C}_j|t}} & \text{if } r\gg 1,\end{cases} \tag{106}$$

where $Q_1(u_j,\sigma)$ is a function explicitly determined by the processes which occur in the system under consideration.

(1c) For $|B_j|\neq 2|\mathcal{C}_j|$, $|\gamma_{j'\neq j}|=|B_j|-2|\mathcal{C}_j|$, $|\gamma_j|>0$, and $\mathcal{D}^j_{1,2}+2(\text{sign}\,\mathcal{C}_j)\mathcal{D}^j_{2,2}\neq 0$, we have

$$\mathcal{G}^{jj}_r(t)-\mathcal{G}^{jj}_r(\infty)\sim\begin{cases}e^{-|\gamma_j|t} & \text{if } \varphi_{jj}=|\gamma_j|\\[1ex] e^{-|\gamma_{j'\neq j}|t}\sqrt{\dfrac{t}{\pi|\mathcal{C}_j|}} & \text{if } \varphi_{jj}=|\gamma_{j'\neq j}|.\end{cases} \tag{107}$$

(1d) For $|B_j|\neq 2|\mathcal{C}_j|$, $|\gamma_j|=|B_j|-2|\mathcal{C}_j|$, $|\gamma_{j'\neq j}|>0$, and $\mathcal{D}^j_{1,1}+2(\text{sign}\,\mathcal{C}_j)\mathcal{D}^j_{2,1}\neq 0$, we have

$$\mathcal{G}^{jj}_r(t)-\mathcal{G}^{jj}_r(\infty)\sim e^{-|\gamma_j|t}\sqrt{\frac{t}{\pi|\mathcal{C}_j|}}. \tag{108}$$

The correlation functions $\mathcal{G}^{AB}_r(t)$ have to be discussed separately. With the definition

$$\varphi_{AB}\equiv\min(|\gamma_A|,|\gamma_B|,|B_{AB}|-2|\mathcal{C}_{AB}|), \tag{109}$$

(1e) for $\varphi_{AB}=|\gamma_A|\neq|B_j|-2|\mathcal{C}_j|$, we have

$$\mathcal{G}^{AB}_r(t)-\mathcal{G}^{AB}_r(\infty)\sim e^{-|\gamma_A|t}. \tag{110}$$

(1f) For $\varphi_{AB}=|\gamma_B|\neq|B_{AB}|-2|\mathcal{C}_{AB}|$, we have

$$\mathcal{G}^{AB}_r(t)-\mathcal{G}^{AB}_r(\infty)\sim e^{-|\gamma_B|t}. \tag{111}$$

(1g) For $\varphi_{AB}=|B_{AB}|-2|\mathcal{C}_{AB}|>0$ and $\mathcal{G}^{AB}_{|r|>0}\neq 0$, we have

$$\mathcal{G}^{AB}_r(t)-\mathcal{G}^{AB}_r(\infty)\sim\rho_A(0)\rho_B(0)e^{-(|B_{AB}|-2|\mathcal{C}_{AB}|)t}. \tag{112}$$

Notice again that if $\mathcal{G}^{AB}_r(0)=0,\forall r$, we arrive at

$$\mathcal{G}^{AB}_r(t)-\mathcal{G}^{AB}_r(\infty)\sim\begin{cases}\dfrac{Q_1(u_j,\sigma)e^{-(|B_{AB}|-2|\mathcal{C}_{AB}|)t}}{\sqrt{4\pi|\mathcal{C}_{AB}|t}} & \text{if } r\ll L\\[2ex] \dfrac{Q_1(u_{AB},\sigma)e^{-\sigma^2u_{AB}-(|B_{AB}|-2|\mathcal{C}_{AB}|)t}}{\sqrt{4\pi|\mathcal{C}_{AB}|t}} & \text{if } r\gg 1,\end{cases} \tag{113}$$

where $Q_1(u_j,\sigma)$ is the same quantity as above.

(1h) For $|B_{AB}|\neq 2|\mathcal{C}_{AB}|$, $|\gamma_A|=|B_{AB}|-2|\mathcal{C}_{AB}|$, and $\mathcal{D}^{AB}_{1,1}+2(\text{sign}\,\mathcal{C}_{AB})\mathcal{D}^{AB}_{2,1}\neq 0$, we have

$$\mathcal{G}_r^{AB}(t)-\mathcal{G}_r^{AB}(\infty)\sim\begin{cases} e^{-|\gamma_B|t} & \text{if } \varphi_{AB}=|\gamma_B| \\ e^{-|\gamma_A|t}\sqrt{\dfrac{t}{\pi|\mathcal{C}_{AB}|}} & \text{if } \varphi_{AB}=|\gamma_A|. \end{cases} \tag{114}$$

(1i) For $|B_{AB}|\neq 2|\mathcal{C}_{AB}|$, $|\gamma_B|=|B_{AB}|-2|\mathcal{C}_{AB}|$, and $\mathcal{D}_{1,2}^{AB}+2(\operatorname{sign}\mathcal{C}_{AB})\mathcal{D}_{2,2}^{AB}\neq 0$, we have

$$\mathcal{G}_r^{AB}(t)-\mathcal{G}_r^{AB}(\infty)\sim\begin{cases} e^{-|\gamma_A|t} & \text{if } \varphi_{AB}=|\gamma_A| \\ e^{-|\gamma_B|t}\sqrt{\dfrac{t}{\pi|\mathcal{C}_{AB}|}} & \text{if } \varphi_{AB}=|\gamma_B|. \end{cases} \tag{115}$$

(2) For $|B_l|=2|\mathcal{C}_l|$, the correlation functions decay algebraically. It is appropriate to distinguish, $r\ll L$ and $r\sim L$. Again, $\sigma=r/L$, $l\in(AA,BB,AB)$.

In the regime $r\ll L$,

$$\mathcal{G}_r^{l}(t)-\mathcal{G}_r^{l}(\infty)\sim\begin{cases} \dfrac{\mathcal{F}_1(u_l,\sigma,\nu)}{(4|\mathcal{C}_l|t)^{\nu/2}} & \text{if } 0<\nu<1 \\ \dfrac{[2\zeta(\nu)+(4\sigma^2u_l|\mathcal{C}_l|t)^{(1-\nu)/2}]}{(4\pi|\mathcal{C}_l|t)^{1/2}} & \text{if } \nu>1, \\ \dfrac{\ln[4|\mathcal{C}_l|u_l(1-\sigma)t]}{(4\pi|\mathcal{C}_l|t)^{1/2}} & \text{if } \nu=1, \end{cases} \tag{116}$$

where the following auxiliary function has been used:

$$\mathcal{F}_1(u,\sigma,\nu)\equiv\frac{\left[\Gamma\left(\dfrac{1-\nu}{2}\right)+\Gamma\left(\dfrac{1-\nu}{2},\sigma^2u_l\right)-\Gamma\left(\dfrac{1-\nu}{2},u_l(1-\sigma)^2\right)-\Gamma\left(\dfrac{1-\nu}{2},u_l(1+\sigma)^2\right)\right]}{\sqrt{4\pi}}, \tag{117}$$

while for $r\gg 1, r\equiv\sigma L\sim L$, we find

$$\mathcal{G}_r^{l}(t)-\mathcal{G}_r^{l}(\infty)\sim\begin{cases} \dfrac{\mathcal{F}_2(u_l,\sigma,\nu)}{(4|\mathcal{C}_l|t)^{\nu/2}} & \text{if } 0<\nu<1 \\ \dfrac{[(1+e^{-\sigma^2u_l})\zeta(\nu)+((1-\sigma)/\sigma)(4\sigma^2u_l|\mathcal{C}_l|t)^{(1-\nu)/2}]}{(4\pi|\mathcal{C}_l|t)^{1/2}} & \text{if } \nu>1 \\ \dfrac{\ln(4|\mathcal{C}_l|u_lt)}{(4\pi|\mathcal{C}_l|t)^{1/2}}\,e^{-\sigma^2u_l} & \text{if } \nu=1, \end{cases} \tag{118}$$

with the auxiliary function

$$\mathcal{F}_2(u_l,\sigma,\nu)\equiv\frac{e^{-\sigma^2u_l}}{(1-\nu)}\sqrt{\frac{u_l\sigma^2}{\pi}}. \tag{119}$$

Notice that when the initial correlations are absent ($\nu=0$),

$$\mathcal{G}_r^{l}(t)-\mathcal{G}_r^{l}(\infty)\sim\begin{cases} \dfrac{Q_1(u,\sigma)+\mathcal{O}(1)}{\sqrt{4\pi|\mathcal{C}_l|t}} & \text{if } \mathcal{C}_l<0 \\ \dfrac{Q_1(u,\sigma)}{\sqrt{4\pi|\mathcal{C}_l|t}}+\mathcal{F}_1(u,\sigma,\nu)\dfrac{\rho_i(0)\rho_j(0)(1-|\kappa_l|)}{8|\mathcal{C}_l|t} & \text{if } r\ll L \text{ and } \mathcal{C}_l>0 \\ \dfrac{e^{-\sigma^2u_l}Q_1(u,\sigma)}{\sqrt{4\pi|\mathcal{C}_l|t}}+\mathcal{F}_2(u,\sigma,\nu)\dfrac{\rho_i(0)\rho_j(0)(1-|\kappa_l|)}{8|\mathcal{C}_l|t} & \text{if } r\gg 1 \text{ and } \mathcal{C}_l>0, \end{cases} \tag{120}$$

where $Q_1(u,\sigma)$ has been defined previously.

From the above we infer that the initial conditions affect the asymptotic behavior of correlation functions only when the latter decay algebraically (116), (118), and (120). Provided the initial correlations are dominant ($0<\nu_l<1$), the critical exponent is renormalized, while for weak initial correlations ($\nu_l>1$), the exponent is independent of initial correlations, i.e., 1/2. The intermediate regime, $\nu_l=1$, exhibits logarithmic dependence consistent with a marginal behavior.

B. Two-point instantaneous correlation functions on $V_2(d)$ in arbitrary dimension

This section is devoted to the computation of correlation functions in arbitrary dimensions ($d\geqslant 2$) on the manifold $V_2(d)$.

With the notations $r=(r_1,\ldots,r_\alpha,\ldots,r_d)$, $|r|=\sqrt{\Sigma_\alpha r_\alpha^2}$ (sometimes denoted by r, for notational simplicity) and $r_\alpha^\pm \equiv \sqrt{(r_\alpha\pm 1)^2+\Sigma_{\alpha\neq\alpha'} r_{\alpha'}^2}$, solving the multidimensional equations of motion of the correlation functions (see Appendix D), we arrive at the following explicit forms:

$$\begin{aligned}
&\mathcal{G}^{AA}_{|r|=|(r_1,\ldots,r_d)|>0}(t)-[\rho_A(t)]^2\\
&=-[\rho_A(0)]^2e^{-2|\gamma_A|dt}+\left(\frac{\mathcal{C}_A\rho_A(0)+\mathcal{D}^A_{2,0}+\mathcal{D}^A_{2,1}+\mathcal{D}^A_{2,2}}{\mathcal{C}_A}\right)\prod_{\alpha=1,\ldots,d}[e^{-|B_A|t}I_{r_\alpha}(2\mathcal{C}_At)]\\
&\quad+\sum_{r'\neq 0}\mathcal{G}^{AA}_{|r'|}(0)\prod_{\alpha=1,\ldots,d}[e^{-|B_A|t}I_{r_\alpha-r'_\alpha}(2\mathcal{C}_At)]+\left(\mathcal{D}^A_{1,0}+\frac{\mathcal{D}^A_{2,0}|B_A|d}{\mathcal{C}_A}\right)\int_0^t d\tau\prod_{\alpha=1,\ldots,d}[e^{-|B_A|t}I_{r_\alpha}(2\mathcal{C}\tau)]\\
&\quad+\left(\mathcal{D}^A_{1,1}+d\,\frac{\mathcal{D}^A_{2,1}(|B_A|-|\gamma_A|)}{\mathcal{C}_A}\right)e^{-d|\gamma_A|t}\int_0^t d\tau\, e^{-d(|B_A|-|\gamma_A|)\tau}\prod_{\alpha=1,\ldots,d}I_{r_\alpha}(2\mathcal{C}_A\tau)\\
&\quad+d\mathcal{D}^A_{2,2}e^{-d|\gamma_B|t}\left(\frac{|B_A|-|\gamma_B|}{\mathcal{C}_A}\right)\int_0^t d\tau\, e^{-d(|B_A|-|\gamma_B|)\tau}\prod_{\alpha=1\ldots d}I_{r_\alpha}(2\mathcal{C}_A\tau)\\
&\quad-\int_0^t dt'\,\mathcal{G}_1(t')e^{-d|B_A|(t-t')}\left(2\mathcal{C}_Ad\prod_{\alpha=1,\ldots,d}I_{r_\alpha(2\mathcal{C}_A(t-t'))}+\frac{G_1^a-B_A/2}{\mathcal{C}_A}\frac{\partial}{\partial t'}\prod_{\alpha=1,\ldots,d}I_{r_\alpha}[2\mathcal{C}_A(t-t')]\right),
\end{aligned}\tag{121}$$

we also have

$$\begin{aligned}
&\mathcal{G}^{BB}_{|r|=|(r_1,\ldots,r_d)|>0}(t)-[\rho_B(t)]^2\\
&=-[\rho_B(0)]^2e^{-2|\gamma_B|dt}+\left(\frac{\mathcal{C}_B\rho_A(0)+\mathcal{D}^B_{2,0}+\mathcal{D}^B_{2,1}+\mathcal{D}^B_{2,2}}{\mathcal{C}_B}\right)\prod_{\alpha=1,\ldots,d}[e^{-|B_B|t}I_{r_\alpha}(2\mathcal{C}_Bt)]\\
&\quad+\sum_{r'\neq 0}\mathcal{G}^{BB}_{|r'|}(0)\prod_{\alpha=1,\ldots,d}[e^{-|B_B|t}I_{r_\alpha-r'_\alpha}(2\mathcal{C}_Bt)]\\
&\quad+\left(\mathcal{D}^B_{1,0}+\frac{\mathcal{D}^B_{2,0}|B_B|d}{\mathcal{C}_B}\right)\int_0^t d\tau\prod_{\alpha=1,\ldots,d}[e^{-|B_B|t}I_{r_\alpha}(2\mathcal{C}_B\tau)]\\
&\quad+\left(\mathcal{D}^B_{1,1}+d\,\frac{\mathcal{D}^B_{2,1}(|B_B|-|\gamma_B|)}{\mathcal{C}_B}\right)e^{-d|\gamma_B|t}\int_0^t d\tau\, e^{-d(|B_B|-|\gamma_B|)\tau}\prod_{\alpha=1,\ldots,d}I_{r_\alpha}(2\mathcal{C}_B\tau)\\
&\quad+d\mathcal{D}^B_{2,2}e^{-d|\gamma_A|t}\left(\frac{|B_B|-|\gamma_A|}{\mathcal{C}_B}\right)\int_0^t d\tau\, e^{-d(|B_B|-|\gamma_A|)\tau}\prod_{\alpha=1,\ldots,d}I_{r_\alpha}(2\mathcal{C}_B\tau)\\
&\quad-\int_0^t dt'\,\mathcal{G}_1(t')e^{-d|B_B|(t-t')}\left(2\mathcal{C}_Bd\prod_{\alpha=1,\ldots,d}I_{r_\alpha[2\mathcal{C}_B(t-t')]}+\frac{G_2^b-B_B/2}{\mathcal{C}_B}\frac{\partial}{\partial t'}\prod_{\alpha=1,\ldots,d}I_{r_\alpha}[2\mathcal{C}_B(t-t')]\right),
\end{aligned}\tag{122}$$

and

$$\mathcal{G}^{AB}_{|r|=|(r_1,\ldots,r_d)|>0}(t)-[\rho_A(t)\rho_B(t)]\tag{123}$$

$$\begin{aligned}
&=-[\rho_A(\infty)\rho_B(0)+\rho_A(0)\rho_B(\infty)-\rho_A(\infty)\rho_B(\infty)]e^{-|\gamma_A+\gamma_B|dt}-[\rho_A(0)\rho_B(0)]e^{-d(|\gamma_A|+|\gamma_B|)t}\\
&\quad+\left(\frac{\mathcal{D}^{AB}_{2,0}+\mathcal{D}^{AB}_{2,1}+\mathcal{D}^{AB}_{2,2}}{\mathcal{C}_{AB}}\right)\prod_{\alpha=1,\ldots,d}[e^{-|B_{AB}|t}I_{r_\alpha}(2\mathcal{C}_{AB}t)]+\sum_{r'\neq 0}\mathcal{G}^{AB}_{|r'|}(0)\prod_{\alpha=1,\ldots,d}[e^{-|B_{AB}|t}I_{r_\alpha-r'_\alpha}(2\mathcal{C}_{AB}t)]\\
&\quad+\left(\mathcal{D}^{AB}_{1,0}+\frac{\mathcal{D}^{AB}_{2,0}|B_{AB}|d}{\mathcal{C}_{AB}}\right)\int_0^t d\tau\prod_{\alpha=1,\ldots,d}[e^{-|B_{AB}|t}I_{r_\alpha}(2\mathcal{C}_{AB}\tau)]
\end{aligned}$$

$$+\left(\mathcal{D}_{1,1}^{AB}+d\,\frac{\mathcal{D}_{2,1}^{AB}(|B_{AB}|-|\gamma_A|)}{\mathcal{C}_{AB}}\right)e^{-d|\gamma_A|t}\int_0^t d\tau\, e^{-d(|B_{AB}|-|\gamma_A|)\tau}\prod_{\alpha=1,\dots,d} I_{r_\alpha}(2\mathcal{C}_{AB}\tau)$$

$$+\left[\mathcal{D}_{1,2}^{AB}+d\mathcal{D}_{2,2}^{AB}\left(\frac{|B_{AB}|-|\gamma_B|}{\mathcal{C}_{AB}}\right)\right]e^{-d|\gamma_B|t}\int_0^t d\tau\, e^{-d(|B_{AB}|-|\gamma_B|)\tau}\prod_{\alpha=1\dots d} I_{r_\alpha}(2\mathcal{C}_{AB}\tau)$$

$$-\int_0^t dt'\,\mathcal{G}_1(t')e^{-d|B_{AB}|(t-t')}\left(2\mathcal{C}_{AB}d\prod_{\alpha=1,\dots,d} I_{r_\alpha}[2\mathcal{C}_{AB}(t-t')]\right.$$

$$\left.+\frac{H_1^{ab}+H_2^{ab}-A_1^a-D_2^b}{\mathcal{C}_{AB}}\,\frac{\partial}{\partial t'}\prod_{\alpha=1\dots d} I_{r_\alpha}[2\mathcal{C}_{AB}(t-t')]\right).$$

Notice that these expression are valid, on $V_2(d)$, in arbitrary dimension and setting $d=1$ we recover the one-dimensional expressions of the preceding section.

We assume here that the initial state is characterized by a random, translationally invariant, but uncorrelated initial distribution, i.e., $\mathcal{G}^l_{|r|>0}(0)=\rho_i(0)\rho_j(0)$.

The asymptotic behavior is obtained similarly to the one-dimensional case when $|\mathcal{C}_l|t\gg 1$ with $u_l=L^2/4|\mathcal{C}_l|t$.

(1) For $|B_l|>2|\mathcal{C}_l|$, the decay of the correlation function is exponential,

$$\varphi_{jj}\equiv\min(|\gamma_j|,|B_j|-2|\mathcal{C}_j|). \tag{124}$$

(1a) If $\varphi_{jj}=|\gamma_A|\neq|B_j|-2|\mathcal{C}_j|$, we have

$$\mathcal{G}_r^{jj}(t)-\mathcal{G}_r^{jj}(\infty)\sim e^{-d|\gamma_j|t}. \tag{125}$$

(1b) If $\varphi_{jj}=|B_j|-2|\mathcal{C}_j|>|\gamma_j|$ and $\mathcal{G}^{jj}_{r\neq 0}(0)\neq 0$, we have

$$\mathcal{G}_r^{jj}(t)-\mathcal{G}_r^{jj}(\infty)\sim e^{-d(|B_j|-2|\mathcal{C}_j|)t}. \tag{126}$$

Finally, note that provided $\mathcal{G}_{r\neq 0}(0)=0$, we find

$$\mathcal{G}_r^{jj}(t)-\mathcal{G}_r^{jj}(\infty)\sim\begin{cases}\dfrac{Q_2(u_j,\sigma_\alpha)e^{-(|B_j|-2|\mathcal{C}_j|)t}}{(4\pi|\mathcal{C}_j|t)^{d/2}} & \text{if } r\ll L\\[2ex] \dfrac{Q_2(u_j,\sigma_\alpha)\exp\left(-\sum_\alpha \sigma_\alpha^2 u_j-(|B_j|-2|\mathcal{C}_j|)t\right)}{(4\pi|\mathcal{C}_j|t)^{d/2}} & \text{if } r\gg 1,\end{cases} \tag{127}$$

where $Q_2(u_j,\sigma_\alpha)$ is a known function determined by the processes occurring in the model under consideration.

(1c) If $|B_j|\neq 2|\mathcal{C}_j|$, $|\gamma_{j'\neq æ}|=|B_j|-2|\mathcal{C}_j|$, $|\gamma_j|>0$, and $\mathcal{D}^j_{1,2}+2(\operatorname{sign}\mathcal{C}_j)\mathcal{D}^j_{2,2}\neq 0$, we have

$$\mathcal{G}_r^{jj}(t)-\mathcal{G}_r^{jj}(\infty)\sim\begin{cases} e^{-d|\gamma_j|t} & \text{if } \varphi_{jj}=|\gamma_j|\\ e^{-|\gamma_{j'\neq j}|t}\sqrt{\dfrac{t}{\pi|\mathcal{C}_j|}} & \text{if } \varphi_{jj}=|\gamma_{j'\neq j}|\\ e^{-2|\gamma_{j'\neq j}|t}\ln t & \text{if } \varphi_{jj}=|\gamma_{j'\neq j}| \text{ and } d=2\\ e^{-d|\gamma_{j'\neq j}|t}(4\pi|\mathcal{C}_j|t)^{1-d/2} & \text{if } \varphi_{jj}=|\gamma_{j'\neq j}| \text{ and } d\geq 3.\end{cases} \tag{128}$$

(1d) If $|B_j|\neq 2|\mathcal{C}_j|$, $|\gamma_j|=|B_j|-2|\mathcal{C}_j|$, $|\gamma_{j'\neq j}|>0$, and $\mathcal{D}^{\,j}_{1,1}+2(\operatorname{sign}\mathcal{C}_j)\mathcal{D}^{\,j}_{2,1}\neq 0$, we have

$$\mathcal{G}_r^{jj}(t)-\mathcal{G}_r^{jj}(\infty)\sim\begin{cases} e^{-|\gamma_j|t}\sqrt{\dfrac{t}{\pi|\mathcal{C}_j|}} & \text{if } d=1\\ e^{-2|\gamma_j|t}\ln t & \text{if } d=2\\ e^{-d|\gamma_j|t}(4\pi|\mathcal{C}_j|t)^{1-d/2} & \text{if } d\geq 3.\end{cases} \tag{129}$$

The functions $\mathcal{G}_r^{AB}(t)$ requires a separate discussion. With the definition of φ_{AB}

$$\varphi_{AB}\equiv\min(|\gamma_A|,|\gamma_B|,|B_{AB}|-2|\mathcal{C}_{AB}|).$$

(1e) If $\varphi_{AB}=|\gamma_A|\neq|B_j|-2|\mathcal{C}_j|$, we have

$$\mathcal{G}_r^{AB}(t)-\mathcal{G}_r^{AB}(\infty)\sim e^{-d|\gamma_A|t}. \tag{130}$$

(1f) If $\varphi_{AB}=|\gamma_B|\neq|B_{AB}|-2|\mathcal{C}_{AB}|$, we have

$$\mathcal{G}_r^{AB}(t)-\mathcal{G}_r^{AB}(\infty)\sim e^{-d|\gamma_B|t}. \tag{131}$$

(1g) If $\varphi_{AB}=|B_{AB}|-2|\mathcal{C}_{AB}|\neq|\gamma_{A,B}|>0$ and $\mathcal{G}_{r\neq 0}(0)\neq 0$, we have

$$\mathcal{G}_r^{AB}(t)-\mathcal{G}_r^{AB}(\infty)\sim e^{-d(|B_{AB}|-2|\mathcal{C}_{AB}|)t}. \tag{132}$$

Again, when $\mathcal{G}_{r\neq 0}^{AB}(0)=0,\forall r$, we arrive at

$$\mathcal{G}_r^{AB}(t)-\mathcal{G}_r^{AB}(\infty)\sim\begin{cases}\dfrac{Q_2(u_{AB},\sigma_\alpha)e^{-d(|B_{AB}|-2|\mathcal{C}_{AB}|)t}}{(4\pi|\mathcal{C}_{AB}|t)^{d/2}} & \text{if } r\ll L\\[2ex] \dfrac{Q_2(u_{AB},\sigma_\alpha)e^{-du_{AB}-d(|B_{AB}|-2|\mathcal{C}_{AB}|)t}}{(4\pi|\mathcal{C}_{AB}|t)^{d/2}} & \text{if } r\gg 1,\end{cases} \tag{133}$$

where $Q_2(u_{AB},\sigma_\alpha)$ has been defined previously.

(1h) If $|B_{AB}|\neq 2|\mathcal{C}_{AB}|$, $|\gamma_A|=|B_{AB}|-2|\mathcal{C}_{AB}|$, and $\mathcal{D}_{1,1}^{AB}+2(\mathrm{sign}\,\mathcal{C}_{AB})\mathcal{D}_{2,1}^{AB}\neq 0$, we have

$$\mathcal{G}_r^{AB}(t)-\mathcal{G}_r^{AB}(\infty)\sim\begin{cases}e^{-d|\gamma_B|t} & \text{if } \varphi_{AB}=|\gamma_B|\\ e^{-|\gamma_A|t}\sqrt{\dfrac{t}{\pi|\mathcal{C}_{AB}|}} & \text{if } \varphi_{AB}=|\gamma_A| \text{ and } d=1\\ e^{-2|\gamma_A|t}\ln t & \text{if } \varphi_{AB}=|\gamma_A| \text{ and } d=2\\ e^{-d|\gamma_A|t}(4\pi|\mathcal{C}_{AB}|t)^{1-d/2} & \text{if } \varphi_{AB}=|\gamma_A| \text{ and } d\geq 3.\end{cases} \tag{134}$$

(1i) If $|B_{AB}|\neq 2|\mathcal{C}_{AB}|$, $|\gamma_B|=|B_{AB}|-2|\mathcal{C}_{AB}|$, and $\mathcal{D}_{1,2}^{AB}+2(\mathrm{sign}\,\mathcal{C}_{AB})\mathcal{D}_{2,2}^{AB}\neq 0$, we have

$$\mathcal{G}_r^{AB}(t)-\mathcal{G}_r^{AB}(\infty)\sim\begin{cases}e^{-d|\gamma_A|t} & \text{if } \varphi_{AB}=|\gamma_A|\\ e^{-|\gamma_B|t}\sqrt{\dfrac{t}{\pi|\mathcal{C}_{AB}|}} & \text{if } \varphi_{AB}=|\gamma_B| \text{ and } d=1\\ e^{-|\gamma_B|t}\ln t & \text{if } \varphi_{AB}=|\gamma_B| \text{ and } d=2\\ e^{-|\gamma_B|t}(4\pi|\mathcal{C}_{AB}t|^{1-d/2}) & \text{if } \varphi_{AB}=|\gamma_B| \text{ and } d\geq 3.\end{cases} \tag{135}$$

(2) For $|B_l|=2|\mathcal{C}_l|$, the decay of correlation functions is algebraic. We distinguish the regime $r\ll L$, from that where $r\sim L$ $(\sigma=r/L,\ l\in(AA,BB,AB))$.

In the limit $r\ll L$,

$$\mathcal{G}_r^{l}(t)-\mathcal{G}_r^{l}(\infty)\sim\begin{cases}\dfrac{e\sum_\alpha \sigma_\alpha^2 u\,\rho_i(0)\rho_j(0)}{(4\pi|\mathcal{C}_l|t)^{d/2}} & \text{if } \mathcal{C}_l<0\\[2ex] \dfrac{Q_2(u_l,\sigma_\alpha)}{(4\pi|\mathcal{C}_l|t)^{d/2}}+\dfrac{\rho_i(0)\rho_j(0)\prod_{\alpha=1,\dots,d}\mathcal{F}_{1,\alpha}(u_l,\sigma_\alpha,\nu=0)}{8|\mathcal{C}_l|t} & \text{if } \mathcal{C}_l>0,\end{cases} \tag{136}$$

with the definition of the following auxiliary functions:

$$\mathcal{F}_{1,\alpha}(u,\sigma_\alpha,\nu)\equiv\frac{\left[\Gamma\left(\frac{1-\nu}{2}\right)+\Gamma\left(\frac{1-\nu}{2},\sigma_\alpha^2u_l\right)-\Gamma\left(\frac{1-\nu}{2},u_l(1-\sigma_\alpha)^2\right)-\Gamma\left(\frac{1-\nu}{2},u_l(1+\sigma_\alpha)^2\right)\right]}{\sqrt{4\pi}}. \tag{137}$$

In the regime where $r\gg 1, r_\alpha\equiv\sigma_\alpha L\sim L$, we find

$$\mathcal{G}_r^{l}(t)-\mathcal{G}_r^{l}(\infty)\sim\begin{cases}\dfrac{e\sum_\alpha \sigma_\alpha^2 u\,\rho_i(0)\rho_j(0)}{(4\pi|\mathcal{C}_l|t)^{d/2}} & \text{if } \mathcal{C}_l<0\\[2ex] \dfrac{Q_2(u_l,\sigma_\alpha)}{(4\pi|\mathcal{C}_l|t)^{d/2}}+\dfrac{\rho_i(0)\rho_j(0)\prod_{\alpha=1,\dots,d}\mathcal{F}_{2,\alpha}(u_l,\sigma_\alpha,\nu=0)}{8|\mathcal{C}_l|t} & \text{if } \mathcal{C}_l>0,\end{cases} \tag{138}$$

with the auxiliary functions defined as

$$\mathcal{F}_{2,\alpha}(u_l,\sigma_\alpha,\nu)\equiv\frac{e^{-\sigma_\alpha^2u_l}}{1-\nu}\sqrt{\frac{u_l\sigma_\alpha^2}{\pi}}. \tag{139}$$

Notice that setting $\mathcal{G}_r(0)=0,\forall r$, leads to

$$\mathcal{G}_r^l(t)-\mathcal{G}_r^l(\infty)\sim\begin{cases}\dfrac{Q_2(u_l,\sigma_\alpha)}{(4\pi|\mathcal{C}_j t|)^{d/2}} & \text{if } r\ll L\\ \dfrac{Q_2(u_l,\sigma_\alpha)e^{-\sum_\alpha \sigma_\alpha^2 u_l}}{(4\pi|\mathcal{C}_j t|)^{d/2}} & \text{if } r\gg 1,\end{cases} \tag{140}$$

where the function $Q_2(u,\sigma)$ is as above.

We see that for the uncorrelated initial state under consideration, the dimensionality of the problem has a nontrivial effect on the dynamics. In fact, in the critical regime (138) and (140), when $d>2$ the correlation functions decay as t^{-1} instead of $t^{-1/2}$, as in lower dimensions. We further remark that also in the massive case, the dimensionality of the model can have particular nontrivial effects on the asymptotic regime [see, e.g., Eqs. (128), (129), (134), and (135)]. Let us conclude by noting that all results obtained in this section are compatible and in agreement with the previous one-dimensional results.

VI. CONCLUSIONS

In this technical paper, we have classified the solutions of the two-species bimolecular diffusion-limited reaction models and have been able to obtain exact and explicit results, namely the following.

(i) The Fourier-Laplace transform of the density and of the noninstanateous two-point correlation functions (dynamic form factors) on a 56-parameters space, in arbitrary dimensions.

(ii) Exact computation, in arbitrary dimensions, of the density on a 50-parameters manifold for various initial conditions.

Exact computation of the noninstantaneous two-point correlation functions on a 50-parameters manifold for uncorrelated homogeneous, but random, initial states as well as for initially correlated states, in arbitrary dimensions.

(iii) Exact results for the instantaneous two-point correlation functions on a translationally invariant 31-parameters space manifold in arbitrary dimensions.

(iv) Exploring the various classes of solutions for the one- and two-point correlation functions, we have seen in real space and time that there are essentially two regimes, a massive one and an algebraic one for the density and two-point correlation functions. For noninstantaneous correlation function we have pointed out that a drift can occur due to an asymmetry of the reaction-rates characterizing the stochastic Hamiltonian. We have also shown that when initial correlations are strong enough, the critical exponents in the asymptotic regime are renormalized while for weak initial correlations, the long-time behavior is insensitive to and independent of the initial state.

Our approach applied on three-states models (of hard-core particles) is in a sense complementary to previous rigorous results [15] (see the end of Sec. III) and allows to study exactly physical models such as a *three-states* growth model [10]. From our analysis, it follows (alternatively, using the Rauth–Hurwitz conditions and simple algebra) that in arbitrary dimensions, two-species models belonging to the class of soluble models discussed here, do not exhibit phase transitions, nor pattern formation. This has led us to *conjecture* that such a property holds true, in *soluble* models (in the sense discussed here) for an arbitrary number of species s and in arbitrary dimensions.

ACKNOWLEDGMENT

The support of Swiss National Fonds is gratefully acknowledged.

APPENDIX A: DEFINITIONS AND ABBREVIATIONS

In this appendix we introduce the definitions which are adopted throughout the paper. In (19) and in the following, we have used these notations:

$$\begin{gathered}A_0^a\equiv\Gamma_{00}^{10}+\Gamma_{00}^{11}+\Gamma_{00}^{12},\quad A_1^a\equiv\Gamma_{10}^{10}+\Gamma_{10}^{11}+\Gamma_{10}^{12}-A_0^a,\quad A_2^a\equiv\Gamma_{01}^{10}+\Gamma_{01}^{11}+\Gamma_{01}^{12}-A_0^a,\quad B_1^a\equiv\Gamma_{20}^{10}+\Gamma_{20}^{11}+\Gamma_{20}^{12}-A_0^a,\\ B_2^a\equiv\Gamma_{02}^{10}+\Gamma_{02}^{11}+\Gamma_{02}^{12}-A_0^a,\quad C_0^a\equiv\Gamma_{00}^{01}+\Gamma_{00}^{11}+\Gamma_{00}^{21},\quad C_1^a\equiv\Gamma_{10}^{01}+\Gamma_{10}^{11}+\Gamma_{10}^{21}-C_0^a,\\ C_2^a\equiv\Gamma_{01}^{01}+\Gamma_{01}^{11}+\Gamma_{01}^{21}-C_0^a,\quad D_1^a\equiv\Gamma_{20}^{01}+\Gamma_{20}^{11}+\Gamma_{20}^{21}-C_0^a,\quad D_2^a\equiv\Gamma_{02}^{01}+\Gamma_{02}^{11}+\Gamma_{02}^{21}-C_0^a,\quad A_0^b\equiv\Gamma_{00}^{20}+\Gamma_{00}^{21}+\Gamma_{00}^{22},\\ A_1^b\equiv\Gamma_{10}^{20}+\Gamma_{10}^{21}+\Gamma_{10}^{22}-A_0^b,\quad A_2^b\equiv\Gamma_{01}^{20}+\Gamma_{01}^{21}+\Gamma_{01}^{22}-A_0^b,\quad B_1^b\equiv\Gamma_{20}^{20}+\Gamma_{20}^{21}+\Gamma_{20}^{22}-A_0^b,\quad B_2^b\equiv\Gamma_{02}^{20}+\Gamma_{02}^{21}+\Gamma_{02}^{22}-A_0^b,\\ C_0^b\equiv\Gamma_{00}^{02}+\Gamma_{00}^{12}+\Gamma_{00}^{22},\quad C_1^b\equiv\Gamma_{10}^{02}+\Gamma_{10}^{12}+\Gamma_{10}^{22}-C_0^b,\quad C_2^b\equiv\Gamma_{01}^{02}+\Gamma_{01}^{12}+\Gamma_{01}^{22}-C_0^b,\\ D_1^b\equiv\Gamma_{20}^{02}+\Gamma_{20}^{12}+\Gamma_{20}^{22}-C_0^b,\quad D_2^b\equiv\Gamma_{02}^{02}+\Gamma_{02}^{12}+\Gamma_{02}^{22}-C_0^b.\end{gathered} \tag{A1}$$

Further we also use the following notations:

$$E_0^a\equiv\Gamma_{00}^{11},\quad E_0^b\equiv\Gamma_{00}^{22},\quad E_0^{ab}\equiv\Gamma_{00}^{12},\quad E_0^{ba}\equiv\Gamma_{00}^{21},\quad F_1^a\equiv\Gamma_{10}^{11}-\Gamma_{00}^{11},\quad F_1^b\equiv\Gamma_{10}^{22}-\Gamma_{00}^{22}, \tag{A2}$$

$$F_1^{ab}\equiv\Gamma_{10}^{12}-\Gamma_{00}^{12},\quad F_1^{ba}\equiv\Gamma_{10}^{21}-\Gamma_{00}^{21},\quad F_2^a\equiv\Gamma_{01}^{11}-\Gamma_{00}^{11},\quad F_2^b\equiv\Gamma_{01}^{22}-\Gamma_{00}^{22},\quad F_2^{ab}\equiv\Gamma_{01}^{12}-\Gamma_{00}^{12},\quad F_2^{ba}\equiv\Gamma_{01}^{21}-\Gamma_{00}^{21},$$

$$F_3^a\equiv\Gamma_{20}^{11}-\Gamma_{00}^{11},\quad F_3^b\equiv\Gamma_{20}^{22}-\Gamma_{00}^{22},\quad F_3^{ab}\equiv\Gamma_{20}^{12}-\Gamma_{00}^{12},\quad F_3^{ba}\equiv\Gamma_{20}^{21}-\Gamma_{00}^{21},\quad F_4^a\equiv\Gamma_{02}^{11}-\Gamma_{00}^{11},\quad F_4^b\equiv\Gamma_{02}^{22}-\Gamma_{00}^{22},$$

$$F_4^{ab}\equiv\Gamma_{02}^{12}-\Gamma_{00}^{12},\quad F_4^{ba}\equiv\Gamma_{02}^{21}-\Gamma_{00}^{21},\quad G_1^{a}\equiv\Gamma_{00}^{11}+\Gamma_{11}^{11}-\Gamma_{01}^{11}-\Gamma_{10}^{11},\quad G_1^{b}\equiv\Gamma_{00}^{22}+\Gamma_{11}^{22}-\Gamma_{01}^{22}-\Gamma_{10}^{22},$$

$$G_1^{ab}\equiv\Gamma_{00}^{12}+\Gamma_{11}^{12}-\Gamma_{01}^{12}-\Gamma_{10}^{12},\quad G_1^{ba}\equiv\Gamma_{00}^{21}+\Gamma_{11}^{21}-\Gamma_{01}^{21}-\Gamma_{10}^{21},\quad G_2^{a}\equiv\Gamma_{00}^{11}+\Gamma_{22}^{11}-\Gamma_{02}^{11}-\Gamma_{20}^{11},$$

$$G_2^{b}\equiv\Gamma_{00}^{22}-\Gamma_{02}^{22}-\Gamma_{20}^{22}-\Gamma_{22}^{22},\quad G_2^{ab}\equiv\Gamma_{00}^{12}+\Gamma_{22}^{12}-\Gamma_{02}^{12}-\Gamma_{20}^{12},\quad G_2^{ba}\equiv\Gamma_{00}^{21}+\Gamma_{22}^{21}-\Gamma_{02}^{21}-\Gamma_{20}^{21},$$

$$H_1^{a}\equiv\Gamma_{21}^{11}+\Gamma_{00}^{11}-\Gamma_{01}^{11}-\Gamma_{20}^{11},\quad H_2^{a}\equiv\Gamma_{00}^{11}+\Gamma_{12}^{11}-\Gamma_{10}^{11}-\Gamma_{02}^{11},\quad H_1^{b}\equiv\Gamma_{21}^{22}+\Gamma_{00}^{22}-\Gamma_{01}^{22}-\Gamma_{20}^{22},$$

$$H_2^{b}\equiv\Gamma_{00}^{22}+\Gamma_{12}^{22}-\Gamma_{10}^{22}-\Gamma_{02}^{22},\quad H_2^{ab}\equiv\Gamma_{00}^{12}+\Gamma_{12}^{12}-\Gamma_{02}^{12}-\Gamma_{10}^{12},$$

$$H_2^{ba}\equiv\Gamma_{00}^{21}+\Gamma_{12}^{21}-\Gamma_{02}^{21}-\Gamma_{10}^{21},\quad H_1^{ab}\equiv\Gamma_{00}^{12}+\Gamma_{21}^{12}-\Gamma_{20}^{12}-\Gamma_{01}^{12},\quad H_1^{ba}\equiv\Gamma_{00}^{21}+\Gamma_{21}^{21}-\Gamma_{20}^{21}-\Gamma_{01}^{21}.$$

Notice that in the expressions above, Eqs. (A1) and (A2), the rates $\Gamma_{\alpha\beta}^{\alpha\beta}$ have not been made explicit for brevity.

APPENDIX B: DEFINITIONS FOR SEC. V

In Sec. V, in addition to (A1) and (A2), we will also use the following additional definitions :

$$\mathcal{A}_A\equiv 2(A_0^a+C_0^a),\quad B_A\equiv 2(A_1^a+C_2^a),\quad \mathcal{C}_A\equiv A_2^a+C_1^a,$$

$$\mathcal{D}_{1,0}^{A}\equiv\frac{\mathcal{A}_A d}{2}-\left(\frac{B_A}{2}-\mathcal{C}_A+\mathcal{A}_A\right)d\rho_A(\infty),\quad \mathcal{D}_{1,1}^{A}\equiv\left(\frac{B_A}{2}-\mathcal{C}_A+\mathcal{A}_A\right)d[\rho_A(\infty)-\rho_A(0)],\tag{B1}$$

$$\mathcal{D}_{2,0}^{A}\equiv E_0^a+\left(F_1^a+F_2^a-\frac{\mathcal{A}_A}{2}-\mathcal{C}_A\right)\rho_A(\infty)+(F_3^a+F_4^a)\rho_B(\infty),\quad \mathcal{D}_{2,1}^{A}\equiv\left(F_1^a+F_2^a-\frac{\mathcal{A}_A}{2}-\mathcal{C}_A\right)[\rho_A(0)-\rho_A(\infty)],$$

$$\mathcal{D}_{2,2}^{A}\equiv(F_3^a+F_4^a)[\rho_B(0)-\rho_B(\infty)],$$

$$\mathcal{A}_B\equiv 2(A_0^b+C_0^b),\quad B_B\equiv 2(B_1^b+D_2^b),\quad \mathcal{C}_B\equiv B_2^b+D_1^b,$$

$$\mathcal{D}_{1,0}^{B}\equiv\frac{\mathcal{A}_B d}{2}-\left(\frac{B_B}{2}-\mathcal{C}_B+\mathcal{A}_B\right)d\rho_B(\infty),\tag{B2}$$

$$\mathcal{D}_{1,1}^{B}\equiv\left(\frac{B_B}{2}-\mathcal{C}_B+\mathcal{A}_B\right)d[\rho_A(\infty)-\rho_A(0)],$$

$$\mathcal{D}_{2,0}^{B}\equiv E_0^b+(F_1^b+F_2^b)\rho_A(\infty)+\left(F_3^a+F_4^a-\frac{\mathcal{A}_B}{2}-\mathcal{C}_B\right)\rho_B(\infty),\quad \mathcal{D}_{2,1}^{B}\equiv\left(F_3^b+F_4^b-\frac{\mathcal{A}_B}{2}-\mathcal{C}_B\right)[\rho_B(0)-\rho_B(\infty)],$$

$$\mathcal{D}_{2,2}^{B}\equiv(F_1^b+F_2^b)[\rho_A(0)-\rho_A(\infty)],$$

and

$$\mathcal{A}_{AB,1}\equiv\mathcal{A}_B/2,\quad \mathcal{A}_{AB,2}\equiv\mathcal{A}_A/2,\quad B_{AB}\equiv\frac{(B_A+B_B)}{2},\quad \mathcal{C}_{AB}\equiv A_2^a+D_1^b=B_2^b+D_1^b,$$

$$\mathcal{D}_{1,0}^{AB}\equiv-[\mathcal{A}_1^{AB}\rho_A(\infty)+\mathcal{A}_2^{AB}\rho_B(\infty)]d,\quad \mathcal{D}_{1,1}^{AB}\equiv-\mathcal{A}_1^{AB}[\rho_A(0)-\rho_A(\infty)]d,\tag{B3}$$

$$\mathcal{D}_{1,2}^{AB}\equiv-\mathcal{A}_2^{AB}[\rho_B(0)-\rho_B(\infty)]d,\quad \mathcal{D}_{2,0}^{AB}\equiv E_0^{ab}+(F_1^{ab}+F_2^{ab}-C_0^b)\rho_A(\infty)+(F_3^{ab}+F_4^{ab}-A_0^a)\rho_B(\infty),$$

$$\mathcal{D}_{2,1}^{AB}\equiv(F_1^{ab}+F_2^{ab}-C_0^b)[\rho_A(0)-\rho_A(\infty)],\quad \mathcal{D}_{2,2}^{AB}\equiv(F_3^{ab}+F_4^{ab}-C_0^b)[\rho_B(0)-\rho_B(\infty)],$$

where $\rho_A(\infty)=[(A_0^a+C_0^a)/2|\gamma_A|]$ and $\rho_B(\infty)=[(A_0^b+C_0^b)/2|\gamma_B|]$, as in Sec. III.

APPENDIX C: ONE-DIMENSIONAL TWO-POINT CORRELATION FUNCTION ON V_2, THE EQUATIONS OF MOTION AND THEIR SOLUTIONS

In one dimension the equations of motion (96)–(98) of the correlation functions can be written as an unique difference equation:

$$\frac{d}{dt}\mathcal{G}_r^{AA}(t)=B_A\mathcal{G}_r^{AA}(t)+\mathcal{C}_A[\mathcal{G}_{r+1}^{AA}(t)+\mathcal{G}_{r-1}^{AA}(t)]+\mathcal{A}_A\rho_A(t)+(\mathcal{D}_{1,0}^A+\mathcal{D}_{1,1}^A e^{\gamma_A t})\delta_{r,0}+(\mathcal{D}_{2,0}^A+\mathcal{D}_{2,1}^A e^{\gamma_A t}+\mathcal{D}_{2,2}^A e^{\gamma_B t})(\delta_{r,1}+\delta_{r,-1})+[(G_1^a-B_A/2)(\delta_{r,1}+\delta_{r,-1})-2\mathcal{C}_A\delta_{r,0}]\mathcal{G}_1^{AA}(t). \tag{C1}$$

The solution is

$$\begin{aligned}\mathcal{G}_r^{AA}(t)-[\rho_A(t)]^2=&-[\rho_A(0)e^{-|\gamma_A|t}]^2+\rho_A(0)e^{-|B_A|t}I_r(2\mathcal{C}_At)+\sum_{r'\neq 0}\mathcal{G}_{r'}^{AA}(0)e^{-|B_A|t}I_{r-r'}(2\mathcal{C}_At)\\&+\int_0^t dt'\, e^{-|B_A|(t-t')}(\mathcal{D}_{1,0}^A I_r[2\mathcal{C}_A(t-t')]+\mathcal{D}_{2,0}^A\{I_{r+1}[2\mathcal{C}_A(t-t')]+I_{r-1}[2\mathcal{C}_A(t-t')]\})\\&+\mathcal{D}_{1,1}^A\int_0^t dt'\, e^{-|B_A|(t-t')}e^{-|\gamma_A|t'}I_r[2\mathcal{C}_A(t-t')]\\&+\int_0^t dt'\, e^{-|B_A|(t-t')}(\mathcal{D}_{2,1}^A e^{-|\gamma_A|t'}+\mathcal{D}_{2,2}^A e^{-|\gamma_B|t'})\{I_{r+1}[2\mathcal{C}_A(t-t')]+I_{r-1}[2\mathcal{C}_A(t-t')]\}\\&+\int_0^t dt'\, e^{-|B_A|(t-t')}\mathcal{G}_1^{AA}(t')((G_1^a-B_A/2)\{I_{r+1}[2\mathcal{C}_A(t-t')]+I_{r-1}[2\mathcal{C}_A(t-t')]\\&-2\mathcal{C}_AI_r[2\mathcal{C}_A(t-t')]\}).\end{aligned} \tag{C2}$$

Similarly for the B–B correlation functions we get

$$\frac{d}{dt}\mathcal{G}_r^{BB}(t)=B_B\mathcal{G}_r^{BB}(t)+\mathcal{C}_B[\mathcal{G}_{r+1}^{BB}(t)+\mathcal{G}_{r-1}^{BB}(t)]+\mathcal{A}_B\rho_B(t)+(\mathcal{D}_{1,0}^B+\mathcal{D}_{1,1}^B e^{\gamma_B t})\delta_{r,0}+(\mathcal{D}_{2,0}^B+\mathcal{D}_{2,1}^B e^{\gamma_B t}+\mathcal{D}_{2,2}^B e^{\gamma_A t})\times(\delta_{r,1}+\delta_{r,-1})+[(G_2^b-B_B/2)(\delta_{r,1}+\delta_{r,-1})-2\mathcal{C}_B\delta_{r,0}]\mathcal{G}_1^{BB}(t) \tag{C3}$$

and for the A–B correlation function, we have

$$\frac{d}{dt}\mathcal{G}_r^{AB}(t)=B_{AB}\mathcal{G}_r^{AB}(t)+\mathcal{C}_{AB}[\mathcal{G}_{r+1}^{AB}(t)+\mathcal{G}_{r-1}^{AB}(t)]+\mathcal{A}_{AB,1}\rho_A(t)+\mathcal{A}_{AB,2}\rho_B(t)+(\mathcal{D}_{1,0}^{AB}+\mathcal{D}_{1,1}^{AB}e^{\gamma_A t}+\mathcal{D}_{1,2}^{AB}e^{\gamma_B t})\delta_{r,0}+(\mathcal{D}_{2,0}^{AB}+\mathcal{D}_{2,1}^{AB}e^{\gamma_A t}+\mathcal{D}_{2,2}^{AB}e^{\gamma_B t})(\delta_{r,1}+\delta_{r,-1})+[(H_1^{ab}+H_2^{ab}-(A_1^a+D_2^b))(\delta_{r,1}+\delta_{r,-1})-2\mathcal{C}_{AB}\delta_{r,0}]\mathcal{G}_1^{AB}(t). \tag{C4}$$

Equations (C3) and (C4) are solved in a similar way as (C2). The above expressions for $\mathcal{G}_r^{ij}(t),\ (i,j)\in(A,B)$ can be rewritten in a more compact form (99)–(101) using the properties of the modified Bessel functions $I_n(z)$.

APPENDIX D: CORRELATION FUNCTIONS ON V_2 IN ARBITRARY DIMENSION. THE EQUATIONS OF MOTION AND THEIR SOLUTIONS

The equations of motion are the higher dimensional counterparts of the previous equations (C1)–(C4), i.e.,

$$\begin{aligned}\frac{d}{dt}\mathcal{G}_{|r|=|(r_1,\dots,r_d)|}^{AA}(t)=&B_Ad\mathcal{G}_r^{AA}(t)+\mathcal{C}_A\sum_\alpha[\mathcal{G}_{r_\alpha^+}^{AA}(t)+\mathcal{G}_{r_\alpha^-}^{AA}(t)]+d\mathcal{A}_A\rho_A(t)+(\mathcal{D}_{1,0}^A+\mathcal{D}_{1,1}^Ae^{d\gamma_At})\prod_{\alpha=1\dots d}\delta_{r_\alpha,0}\\&+(\mathcal{D}_{2,0}^A+\mathcal{D}_{2,1}^Ae^{d\gamma_At}+\mathcal{D}_{2,2}^Ae^{d\gamma_Bt})\sum_\alpha(\delta_{r_\alpha,e^\alpha}+\delta_{r_\alpha,-e^\alpha})\prod_{\alpha'\neq\alpha}\delta_{r_{\alpha'},0}\\&+\left[(G_1^a-B_A/2)\sum_\alpha(\delta_{r_\alpha,e_\alpha}+\delta_{r_\alpha,-e_\alpha})\prod_{\alpha'\neq\alpha}\delta_{r_{\alpha'},0}-2\mathcal{C}_Ad\prod_{\alpha=1\dots d}\delta_{r_\alpha,0}\right]\mathcal{G}_{|r|=1}^{AA}(t),\end{aligned} \tag{D1}$$

and

$$\frac{d}{dt}\mathcal{G}_{|r|=|(r_1,\dots,r_d)|}^{BB}(t)=B_Bd\mathcal{G}_r^{BB}(t)+\mathcal{C}_B\sum_\alpha[\mathcal{G}_{r_\alpha^+}^{BB}(t)+\mathcal{G}_{r_\alpha^-}^{BB}(t)]+d\mathcal{A}_B\rho_B(t)+(\mathcal{D}_{1,0}^B+\mathcal{D}_{1,1}^Be^{d\gamma_Bt})\prod_{\alpha=1,\dots,d}\delta_{r_\alpha,0}$$

$$+(\mathcal{D}_{2,0}^{B}+\mathcal{D}_{2,1}^{B}e^{d\gamma_B t}+\mathcal{D}_{2,2}^{B}e^{d\gamma_A t})\sum_{\alpha}(\delta_{r_\alpha,e^\alpha}+\delta_{r_\alpha,-e^\alpha})\prod_{\alpha\neq\alpha'}\delta_{r_{\alpha'},0}$$
$$+\left[(G_2^b-B_B/2)\sum_{\alpha}(\delta_{r_\alpha,e_\alpha}+\delta_{r_\alpha,-e_\alpha})\prod_{\alpha\neq\alpha'}\delta_{r_{\alpha'},0}-2\mathcal{C}_B d\prod_{\alpha=1,\ldots,d}\delta_{r_\alpha,0}\right]\mathcal{G}_{|r|=1}^{AA}(t) \tag{D2}$$

and also

$$\frac{d}{dt}\mathcal{G}_{|r|=|(r_1,\ldots,r_d)|}^{AB}(t)=B_{AB}d\mathcal{G}_r^{AB}(t)+\mathcal{C}_{AB}\sum_{\alpha}[\mathcal{G}_{r_\alpha^+}^{AB}(t)+\mathcal{G}_{r_\alpha^-}^{AB}(t)]+d\mathcal{A}_{AB,1}\rho_A(t)+d\mathcal{A}_{AB,2}\rho_B(t)$$
$$+(\mathcal{D}_{1,0}^{AB}+\mathcal{D}_{1,1}^{AB}e^{d\gamma_A t}+\mathcal{D}_{1,2}^{AB}e^{d\gamma_B t})\prod_{\alpha=1,\ldots,d}\delta_{r_\alpha,0}+(\mathcal{D}_{2,0}^{AB}+\mathcal{D}_{2,1}^{AB}e^{d\gamma_A t}+\mathcal{D}_{2,2}^{AB}e^{d\gamma_B t})\sum_{\alpha}(\delta_{r_\alpha,e^\alpha}+\delta_{r_\alpha,-e^\alpha})$$
$$\times\prod_{\alpha\neq\alpha'}\delta_{r_{\alpha'},0}+\left[(H_1^{ab}+H_2^{ab}-A_1^a-D_2^b)\sum_{\alpha}(\delta_{r_\alpha,e_\alpha}+\delta_{r_\alpha,-e_\alpha})\prod_{\alpha\neq\alpha'}\delta_{r_{\alpha'},0}-2\mathcal{C}_{AB}d\prod_{\alpha=1,\ldots,d}\delta_{r_\alpha,0}\right]\mathcal{G}_{|r|=1}^{AB}(t). \tag{D3}$$

The solution of (D1) is

$$\mathcal{G}_{|r|=|(r_1,\ldots,r_d)|}^{AA}(t)-[\rho_A(t)]^2$$
$$=-[\rho_A(0)]^2e^{-2|\gamma_A|dt}+\rho_A(0)e^{-|B_A|dt}\prod_{\alpha=1,\ldots,d}I_{r_\alpha}(2\mathcal{C}_A t)+\sum_{r'\neq 0}\mathcal{G}_{|r'|}^{AA}(0)e^{-|B_A|dt}$$
$$\times\prod_{\alpha=1,\ldots,d}I_{r_\alpha-r'_\alpha}(2\mathcal{C}_A t)+\int_0^t dt'\,e^{-|B_A|d(t-t')}\mathcal{D}_{1,0}^{A}\prod_{\alpha=1,\ldots,d}I_{r_\alpha}[2\mathcal{C}_A(t-t')]+\mathcal{D}_{2,0}^{A}\int_0^t dt'\,e^{-|B_A|d(t-t')}$$
$$\times\sum_{\alpha}\left[\prod_{\alpha'\neq\alpha}I_{r_{\alpha'}}[2\mathcal{C}_A(t-t')]\right](I_{r_\alpha+1}[2\mathcal{C}_A(t-t')]+I_{r_\alpha-1}[2\mathcal{C}_A(t-t')])$$
$$+\int_0^t dt'\,e^{-|B_A|d(t-t')}\mathcal{D}_{1,1}^{A}e^{-|\gamma_A|dt'}\prod_{\alpha=1,\ldots,d}I_{r_\alpha}[2\mathcal{C}_A(t-t')]+\int_0^t dt'\,e^{-|B_A|d(t-t')}$$
$$\times(\mathcal{D}_{2,1}^{A}e^{-|\gamma_A|dt'}+\mathcal{D}_{2,2}^{A}e^{-|\gamma_B|dt'})\sum_{\alpha}\left[\prod_{\alpha'\neq\alpha}I_{r_{\alpha'}}[2\mathcal{C}_A(t-t')]\right][I_{r_\alpha+1}(\cdots)+I_{r_\alpha-1}(\cdots)]$$
$$+\int_0^t dt'\,e^{-|B_A|d(t-t')}\mathcal{G}_{|r|=1}^{AA}(t')(G_1^a-B_A/2)\sum_{\alpha}\left[\prod_{\alpha'\neq\alpha}I_{r_{\alpha'}}[2\mathcal{C}_A(t-t')]\right][I_{r_\alpha+1}(\cdots)+I_{r_\alpha-1}(\cdots)]$$
$$-2\mathcal{C}_A d\int_0^t dt'\,e^{-|B_A|d(t-t')}\mathcal{G}_{|r|=1}^{AA}(t')\prod_{\alpha=1,\ldots,d}e^{-|B_A|d(t-t')}I_{r_\alpha}[2\mathcal{C}_A(t-t')], \tag{D4}$$

where the abbreviated notation $(\cdots)$ instead of $[2\mathcal{C}_A(t-t')]$ has been used. Other correlation functions $\mathcal{G}_{|r|}^{BB}(t)$ and $\mathcal{G}_{|r|}^{AB}(t)$ are obtained in a similar way. Properties of the Bessel functions and elementary manipulations lead to the more compact forms (121)–(123).

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